1. (a) No

Clearly, $(1,0,1) \in W_1$, $(0,1,1) \in W_1$.

But, $(1,0,1)+(0,1,1) = (1,1,2) \notin W_1$.

(b) No

Clearly, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W_2$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in W_2$,

but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin W_2$.

2. (a) No.

$T(i) = i - \bar{i} = 2i$,

$i\,T(i) = -2 \neq T(i \cdot i) = T(-1) = -1 - \overline{-1} = 0$

Remark:

(1) So of you consider two cases: $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$.

But this is not necessary.

The reason is that as stated in the question,

$\mathbb{C}$ is taken to be a one-dimensional complex space here.

If $\mathbb{F} = \mathbb{R}$, the dimension of $\mathbb{C}$ over $\mathbb{R}$ is 2.

(2) For 1(a), 1(b), and 2(a),

the best way to prove some statement is not correct

is to directly give a counter-example.

(b) • $T0 = 0$.

• Let $a = (a_1, a_2, \cdots\cdots)$, $b = (b_1, b_2, \cdots\cdots)$

Then, $Ta + Tb = (a_1 + a_1, a_2 + a_1, \cdots\cdots) + (b_1 + b_1, b_2 + b_1, \cdots\cdots)$

$$= \big((a_1+b_1)+(a_1+b_1), (a_2+b_2)+(a_1+b_1), \cdots\cdots\big)$$

$$= T(a+b)$$

• Let $c \in \mathbb{F}$,

then $cT(a) = c(a_1 + a_1, a_2 + a_1, \cdots\cdots)$

$$= (ca_1 + ca_1, ca_2 + ca_1, \cdots\cdots)$$

$$= T(ca)$$

Hence, T is linear.

3. Suppose $T(x, y, z) = (x_1, y_1, z_1)$.

Then, we know $\left(\frac{x+x_1}{2}, \frac{y+y_1}{2}, \frac{z+z_1}{2}\right)$ is on the plane $x - 2z = 0$,

namely, $\frac{x+x_1}{2} - 2 \cdot \frac{z+z_1}{2} = 0$ — (1)

Also, note that $(2, 0, 1)$, $(2, 1, 1)$ and $(4, 0, 2)$ are all contained by the plane $x - 2z = 0$.

Hence $\big((x, y, z) - (x_1, y_1, z_1)\big) \cdot \big((2,1,1) - (2,0,1)\big) = 0$ — (2)

$\big((x, y, z) - (x_1, y_1, z_1)\big) \cdot \big((4,0,2) - (2,0,1)\big) = 0$ — (3)

Combining (1), (2), and (3),

we see $(x_1, y_1, z_1) = T(x, y, z)$

$$= \left(\frac{3x}{5} + \frac{4z}{5}, y, \frac{4x}{5} - \frac{3z}{5}\right)$$

4. (a) Let $p \in V$.

Then, $T(p) = 0 \iff p(1) = p(2) = \cdots = p(100) = 0$.

So, $p(x) = (x-1)(x-2)\cdots(x-100)\,q(x)$,

where $\deg q(x) \leq 999 - 100 = 899$.

Since $q(x)$ can be arbitrary,

$\dim \ker T = 899 + 1 = 900$.

(b) $\dim \operatorname{range} T = \dim V - \dim \ker T = 100$.

5.

- If $E = \phi$, then the result is trivial.

- If $E \neq \phi$, then there exists some $T \in E$,

s.t., $Tv = w$ for some non-zero $v, w \in V$.

Let $\{v_1, \cdots, v_n\}$ be a basis of V.

Then, for $i = 1, 2, \cdots, n$,

define $R_i(v_j) = \begin{cases} v, & \text{if } j = i \\ 0, & \text{else} \end{cases}$ for $j = 1, \cdots, n$.

$$S_i(u) = \begin{cases} v_i, & \text{if } u = w \\ 0, & \text{else} \end{cases} \quad \text{for arbitrary } u \in V.$$

Hence, $S_i T R_i (v_j) = \begin{cases} v_i, & \text{if } i = j \\ 0, & \text{else} \end{cases}$

Let $T_0 = S_1 T R_1 + S_2 T R_2 + \cdots + S_n T R_n \in E$.

Clearly, T_0 is the identity map I.

Then, for any $S \in \mathcal{L}(V)$,

$S = SI \in E$.

$\Rightarrow E = \mathcal{L}(V)$.