MATH 2050C Mathematical Analysis I
2020-21 Term 2

Solution to Problem Set 8

3.5-2(b)

Denote $x_n = 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$. Given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ satisfying $\frac{1}{2^{N-1}} < \varepsilon$ by Archimedean Property. If $n \geq N$, $\frac{1}{2^{n-1}} \leq \frac{1}{2^{N-1}} < \varepsilon$. For all $n \geq N$ and $\forall k \in \mathbb{N}$, we have

$$|x_{n+k} - x_n| = \frac{1}{(n+1)!} + \cdots + \frac{1}{(n+k)!} < \frac{1}{2^n} + \cdots + \frac{1}{2^{n+k-1}} < \frac{1}{2^{n-1}} < \varepsilon,$$

which verifies the condition of Cauchy sequence.

3.5-5

We have

$$|x_{n+1} - x_n| = \sqrt{n+1} - \sqrt{n} \leq \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2\sqrt{n}}.$$

Since for any $\varepsilon > 0$, we can find $K \in \mathbb{N}$ such that $K > \frac{1}{4\varepsilon^2}$ by Archimedean Property. So for $n \geq K$, we have $|x_{n+1} - x_n| < \frac{1}{\varepsilon}$, which implies

$$\lim |x_{n+1} - x_n| = 0$$

But (x_n) is not a Cauchy sequence since, for a special $\varepsilon = 1$ and any $N \in \mathbb{N}$, we can always choose $n = N$, $k = 3N$. In this case, we have

$$|x_{n+k} - x_n| = \sqrt{4N} - \sqrt{N} = \sqrt{N} \geq 1 = \varepsilon$$

which shows (x_n) is not Cauchy.

3.5-11

Let's show (y_n) is a contractive sequence. For $n > 1$, we have

$$|y_{n+1} - y_n| = \left| \frac{1}{3} y_n + \frac{2}{3} y_{n-1} - y_n \right| = \frac{1}{3} |y_n - y_{n-1}|$$

Hence (y_n) is a contractive sequence and the limit exists.
For the value of $\lim(y_n)$, we note

$$\begin{aligned} y_{n+1} + \frac{2}{3}y_n = \frac{1}{3}y_n + \frac{2}{3}y_{n-1} + \frac{2}{3}y_n &= y_n + \frac{2}{3}y_{n-1} \\ &= y_{n-1} + \frac{2}{3}y_{n-2} \\ &\cdots \\ &= y_2 + \frac{2}{3}y_1 \end{aligned}$$

By taking $n \to \infty$ at left hand side of equation, we have

$$\frac{5}{3}\lim(y_n) = y_2 + \frac{2}{3}y_1 \implies \lim(y_n) = \frac{3}{5}y_2 + \frac{2}{5}y_1$$