MATH 2050C Mathematical Analysis I
2020-21 Term 2

Solution to Problem Set 5

3.2-1

(a) Let $X := (1)$, $Y = \left(\frac{1}{n+1}\right)$, then $\frac{n}{n+1} = X - Y$. Then from Theorem 3.2.3(a), we get

$$\lim(X - Y) = \lim X - \lim Y = 1 - 0 = 1$$

(d) Note that

$$\frac{2n^2+3}{n^2+1} = \frac{2+\frac{3}{n^2}}{1+\frac{1}{n^2}}$$

So

$$\lim\left(\frac{2n^2+3}{n^2+1}\right) = \frac{\lim\left(2+\frac{3}{n^2}\right)}{\lim 1+\frac{1}{n^2}} = \frac{2+\lim\frac{3}{n^2}}{1+\lim\frac{1}{n^2}} = 2$$

3.2-5(b)

Suppose that $((-1)^n n^2)$ is convergent thus bounded. There exists some real number $M > 0$, so that $\left|(-1)^n n^2\right| < M, \forall n \in \mathbb{N}$. Take $N_0 \in \mathbb{N}$ satisfying $N_0 > M + 1$ by the Archimedean Property. Then $(N_0)^2 > (M+1)^2 > M$, contradiction.

3.2-6(d)

Let $X = \left(\frac{n+1}{n}\right)$, $Y = \left(\frac{1}{\sqrt{n}}\right)$, then $\lim X = \lim(1+\frac{1}{n}) = 1$. For the limit $\lim Y$, since for any $\varepsilon > 0$, we can find $K \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon^2}$, then for any $n > K$, we have $\left|0 - \frac{1}{\sqrt{n}}\right| < \varepsilon$, hence $\lim Y = 0$.

So

$$\lim\left(\frac{n+1}{n\sqrt{n}}\right) = \lim X \cdot \lim Y = 0$$

3.2-7

Suppose (b_n) is bounded by M, i.e. $|b_n| \leq M$ for any $n \in \mathbb{N}$. Note that we have

$$-Ma_n \leq a_n b_n \leq Ma_n$$

for any $n \in \mathbb{N}$. By Squeeze Theorem, we have

$$0 = -M \lim(a_n) = \lim(-Ma_n) \leq \lim(a_n b_n) \leq \lim(Ma_n) \leq M \lim(a_n) = 0$$

Hence $\lim(a_n b_n) = 0$.

We cannot directly apply Theorem 3.2.3 since the sequence (b_n) may not have limit.

3.2-9

We only need to find the limit of this sequence. This will show the sequence $(\sqrt{n} y_n)$ converges.

Indeed, we note that

$$\sqrt{n} y_n = \sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \frac{\sqrt{n}(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

We can guess it will take limit $\frac{1}{2}$. Clearly, we will have

$$\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \leq \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n}} = \frac{1}{2}$$

On the other hand, we will have

$$\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \geq \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n+1}} = \frac{1}{2}\sqrt{\frac{n}{n+1}} = \frac{1}{2}\sqrt{1 - \frac{1}{n+1}}$$

Hence, we can apply the operation on these limits to get

$$\lim \frac{1}{n+1} = 0 \implies \lim 1 - \frac{1}{n+1} = 1 \implies \lim \sqrt{\frac{n}{n+1}} = 1 \implies \lim \frac{\sqrt{n}}{2\sqrt{n+1}} = \frac{1}{2}$$

Note that we already have

$$\frac{\sqrt{n}}{2\sqrt{n+1}} \leq \sqrt{n} y_n \leq \frac{1}{2}$$

Then by Squeeze Theorem, we get

$$\lim \sqrt{n} y_n = \frac{1}{2}$$

3.2-18

Let r be a number so that $1 < r < L$ and let $\varepsilon = L - r$. There exists a number $K \in \mathbb{N}$ so that if $n \geq K$ then

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon$$

and

$$\frac{x_{n+1}}{x_n} > L - \varepsilon = r.$$

As $x_n > 0, \forall n \in \mathbb{N}$, $x_{n+K} > r x_{n+K-1} > \cdots > r^n x_K, \forall n \in \mathbb{N}$. Since $r > 1$, for any positive real number M, take n large enough satisfying $r^n > M/x_K$. Thus (x_n) is unbounded and divergent.