MATH 2050C Mathematical Analysis I
2020-21 Term 2

Solution to Problem Set 4

3.1-5

(b) For arbitrary $\varepsilon > 0$, we choose $K > \frac{2}{\varepsilon}$. Then for all $n > K$, we have

$$\left|\frac{2n}{n+1} - 2\right| = \frac{2}{n+1} \leq \frac{2}{K+1} < \frac{2}{K} < \varepsilon$$

Hence $\lim\left(\frac{2n}{n+1}\right) = 2$.

(c) For arbitrary $\varepsilon > 0$, if $K > \frac{13}{4\varepsilon}$, then for all $n > K$,

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \frac{13}{2(2n+5)} < \frac{13}{4n} < \frac{13}{4K} < \varepsilon.$$

3.1-6

(b) For arbitrary $\varepsilon > 0$, we choose $K > \frac{4}{\varepsilon}$. Then for all $n > K$, we have

$$\left|\frac{2n}{n+2} - 2\right| = \frac{4}{n+2} \leq \frac{4}{K+2} < \frac{4}{K} < \varepsilon$$

Hence $\lim\left(\frac{2n}{n+2}\right) = 2$.

(c) For arbitrary $\varepsilon > 0$, if $K > \frac{1}{\varepsilon^2}$, then for all $n > K$,

$$\left|\frac{\sqrt{n}}{n+1}\right| < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{K}} < \frac{1}{\sqrt{\frac{1}{\varepsilon^2}}} = \varepsilon.$$

3.1-9

For any $\varepsilon > 0$, since $\lim(x_n) = 0$, we can find $K \in \mathbb{N}$ such that for any $n > K$, we have

$$|x_n - 0| < \varepsilon^2.$$

Note that $x_n \geq 0$, we actually have $0 \leq x_n < \varepsilon^2$, which implies $0 \leq \sqrt{x_n} < \varepsilon$. So we have

$$|\sqrt{x_n} - 0| < \varepsilon$$

for any $n > K$. Hence $\lim(\sqrt{x_n}) = 0$.

3.1-12

Let's calculate $\sqrt{n^2+1} - n$ first. We have the following

$$\sqrt{n^2+1} - n = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{n}.$$

So for any $\varepsilon > 0$, if we choose $K > \frac{1}{\varepsilon}$ for $K \in \mathbb{N}$, then for any $n > K$, we have

$$\left|\sqrt{n^2+1} - n\right| \leq \frac{1}{n} < \frac{1}{K} < \varepsilon$$

Hence $\lim(\sqrt{n^2+1} - n) = 0$.

3.1-14

We choose $a = \frac{1}{b} - 1$(which implies $b = \frac{1}{1+a}$). Since $0 < b < 1$, we will get $a > 0$. So we have

$$|nb^n - 0| = \frac{n}{(1+a)^n}$$

By the Binomial Theorem,

$$(1+a)^n = 1 + na + \frac{1}{2}n(n-1)a + \cdots \geq \frac{n(n-1)}{2}a^2$$

Hence, we can choose K with $K > \frac{2}{\epsilon a^2} + 1$, then for $n \geq K$, we have

$$|nb^n - 0| \leq \frac{2n}{n(n-1)a^2} = \frac{2}{(n-1)a^2} \leq \frac{2}{(K-1)a^2} < \epsilon$$

This means

$$\lim(nb^n) = 0$$