THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 9 (November 11, 13)

1 Continuous Functions

Definition. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$.

- We say that f is **continuous at** c if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \delta$.
- Let $B \subseteq A$. We say that f is **continuous on** B if f is continuous at every point of B.

Remarks. (1) We do not assume that c is a cluster point of A ($c \in A^c$).

Case 1: If $c \in A^c$, then f is continuous at $c \iff \lim_{x \to c} f = f(c)$.

Case 2: If $c \notin A^c$, then $V_\delta(c) \cap A = \{c\}$ for some $\delta > 0$, so that f is automatically continuous at c.

(2) "f is continuous on B" and "$f\big|_B$ is continuous" are different.

Suppose $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A$.

Sequential Criterion for Continuity. *f is continuous at c if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.*

Discontinuity Criterion. *f is discontinuous at c if and only if there is a sequence (x_n) in A that converges to c but the sequence $(f(x_n))$ does not converge to $f(c)$.*

Example 1. Determine the points of continuity of the function $f(x) := [1/x]$, $x \neq 0$. Here $[\,\cdot\,]$ is the greatest integer function defined by

$$[x] := \sup\{n \in \mathbb{Z} : n \leq x\}.$$

Example 2. Give an example for each of the following:

(a) $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere except at one point.

(b) $f : \mathbb{R} \to \mathbb{R}$ is discontinuous everywhere.

(c) $f : \mathbb{R} \to \mathbb{R}$ is continuous exactly at one point.

(d) $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}\backslash\mathbb{Q}$ but distcontinuous on $\mathbb{Q}$.

Example 3. Let $(r_j)_{j\geq 1}$ be an enumeration of all rational numbers in $\mathbb{R}$. Define the function φ to be

$$\varphi(x) = \sum_{\{j:\, r_j<x\}} \frac{1}{2^j}, \quad x \in \mathbb{R}.$$

Show that

(a) φ is strictly increasing.

(b) φ is discontinuous at every rational numbers.

(c) φ is continuous at every irrational numbers.

Solution. If $y > x$, then

$$\varphi(y) = \sum_{r_j<y} \frac{1}{2^j} = \sum_{r_j<x} \frac{1}{2^j} + \sum_{x\leq r_j<y} \frac{1}{2^j} = \varphi(x) + \sum_{x\leq r_j<y} \frac{1}{2^j} \geq \varphi(x).$$

Hence φ is increasing.

Next we show that φ is discontinuous at rational points. Let x be a rational number, say r_k. Then, for $y > x$,

$$\varphi(y) - \varphi(x) = \sum_{x\leq r_j<y} \frac{1}{2^j} \geq \frac{1}{2^k},$$

which is a constant independent of y. Hence φ is discontinuous at x.

Finally we show that φ is continuous at irrational points. Let x be an irrational number. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $1/2^N < \varepsilon$.

Since $S_N := \{r_j : 1 \leq j \leq N\}$ is finite and $x \notin S_N$, we can find $\delta := \delta_N > 0$ such that

$$(x-\delta, x+\delta) \cap S_N = \emptyset.$$

Now, if $|y - x| < \delta$, then

$$\begin{aligned}
|\varphi(y) - \varphi(x)| &= \left| \sum_{r_j<y} \frac{1}{2^j} - \sum_{r_j<x} \frac{1}{2^j} \right| \\
&\leq \sum_{r_j\in(x-\delta,x+\delta)} \frac{1}{2^j} \\
&\leq \sum_{\substack{r_j\in(x-\delta,x+\delta) \\ j>N}} \frac{1}{2^j} \\
&\leq \sum_{j>N} \frac{1}{2^j} \\
&= \frac{1}{2^N} < \varepsilon.
\end{aligned}$$

Therefore, φ is continuous at x. ◀

Example 4. Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on $\mathbb{Q}$ but distcontinuous on $\mathbb{R}\backslash\mathbb{Q}$?