THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 5 (October 14, 16)

1 The Cauchy Criterion

Definition. A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for any $\varepsilon > 0$ there exists a natural number K such that

$$|x_n - x_m| < \varepsilon \qquad \text{whenever } m, n \geq K.$$

Example 1. Use the definition to determine whether the following sequences are Cauchy.

$$y_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Cauchy Convergence Criterion. *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Definition. A sequence (x_n) of real numbers is said to be **contractive** if there exists a constant r, $0 < r < 1$, such that

$$|x_{n+2} - x_{n+1}| \leq r|x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#}$$

The number r is called the **constant** of the contractive sequence.

Remarks. Do not confuse (#) with the following condition:

$$|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#\#}$$

For example, $(\sqrt{n})$ satisfies (##) but it is not contractive.

Theorem 1. *Every contractive sequence is a Cauchy sequence, and therefore is convergent.*

Example 2. (Sequence of Fibonacci Fractions) Consider the sequence of Fibonacci fractions $x_n := f_n/f_{n+1}$, where (f_n) is the Fibonacci sequence defined by $f_1 = f_2 = 1$ and $f_{n+2} := f_{n+1} + f_n$, $n \in \mathbb{N}$. Show that the sequence (x_n) converges to $1/\varphi$, where $\varphi := (1 + \sqrt{5})/2$ is the Golden Ratio.

Example 3. Let (x_n) be a sequence of real numbers defined by

$$\begin{cases} x_1 = 1, \quad x_2 = 2, \\ x_{n+2} := \dfrac{1}{3}(2x_{n+1} + x_n) \quad \text{for all } n \in \mathbb{N}. \end{cases}$$

Show that (x_n) is convergent and find its limit.

(**Hint:** Note that $x_{n+2} - x_{n+1} = (-\frac{1}{3})(x_{n+1} - x_n)$ and $x_{n+2} - x_1 = \displaystyle\sum_{k=1}^{n+1}(x_{k+1} - x_k)$.)

Solution. Note that

$$x_{n+2} - x_{n+1} = \frac{1}{3}(2x_{n+1} + x_n) - x_{n+1} = -\frac{1}{3}(x_{n+1} - x_n).$$

In particular,

$$|x_{n+2} - x_{n+1}| = \frac{1}{3}|x_{n+1} - x_n| \qquad \text{for } n \in \mathbb{N},$$

so (x_n) is contractive, hence convergent. As

$$x_{n+2} - x_{n+1} = -\frac{1}{3}(x_{n+1} - x_n) = \cdots = (-\frac{1}{3})^n(x_2 - x_1) = (-\frac{1}{3})^n.$$

we have

$$\sum_{k=0}^{n}(x_{k+2} - x_{k+1}) = \sum_{k=0}^{n}(-\frac{1}{3})^k$$
$$x_{n+2} - x_1 = \frac{1 - (-\frac{1}{3})^{n+1}}{1 - (-\frac{1}{3})}.$$

Hence $\lim(x_n) = \lim\left(1 + \frac{3}{4}(1 - (-\frac{1}{3})^{n+1})\right) = \frac{7}{4}$. ◀

2 Infinite Series

Definition. Let (x_n) be a sequence in $\mathbb{R}$. We say that the series $\sum_{n=1}^{\infty} x_n$ is convergent if the sequence of partial sums (s_n):

$$s_n := x_1 + x_2 + \cdots + x_n$$

is convergent.

We say that $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |x_n|$ is convergent.

The nth term test. *If the series $\sum_{n=1}^{\infty} x_n$ converges, then* $\lim(x_n) = 0$.

Example 4. Let (x_n) be a sequence in $\mathbb{R}$. Set

$$r := \limsup |x_n|^{1/n}.$$

Show that

(a) If $r < 1$, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, hence convergent.

(b) If $r > 1$, then $\sum_{n=1}^{\infty} x_n$ is divergent.

Solution. (a) Let $\alpha \in (r, 1)$. Then there is $N \in \mathbb{N}$ such that for $n \geq N$, we have $|x_n|^{1/n} < \alpha$ and hence $|x_n| < \alpha^n$. Thus, for $m > n \geq N$, we have

$$\left|\sum_{k=1}^{m} |x_k| - \sum_{k=1}^{n} |x_k|\right| \leq |x_{n+1}| + \cdots + |x_m| \leq \alpha^{n+1} + \cdots + \alpha^m \leq \frac{\alpha^{n+1}}{1-\alpha}.$$

Since $\lim_n \alpha^n = 0$, $(\sum_{k=1}^{n} |x_k|)$ is Cauchy and hence convergent. Therefore $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

(b) Let $\alpha \in (1, r)$. Then for all $n \in \mathbb{N}$, there is $m \geq n$ such that $|x_m|^{1/m} > \alpha$, so that $|x_m| > \alpha^m \geq 1$. By the nth term test, $\sum_{n=1}^{\infty} x_n$ is divergent.

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