THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 10 (November 18, 20)

1 Uniform Continuity

Definition. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f is **uniformly continuous** on A if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $x, u \in A$ satisfy $|x - u| < \delta(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$.

Nonuniform Continuity Criteria. *Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then the following statements are equivalent:*

(i) f is not uniformly continuous on A.

(ii) There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δ in A such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.

(iii) There exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ but $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Uniform Continuity Theorem. *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.*

Example 1. Determine whether the following functions are uniformly continuous:

(a) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$,

(b) $g : \mathbb{R} \to \mathbb{R}$ definedy by $g(x) = \cos(x^2)$.

Solution. (a) Note that for any $x, y \in [0, \infty)$,

$$\begin{aligned} |f(x) - f(y)|^2 &= |\sqrt{x} - \sqrt{y}|^2 \\ &\leq |\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| \\ &= |x - y|. \end{aligned}$$

Let $\varepsilon > 0$. Take $\delta = \varepsilon^2$. Now, if $x, y \in [0, \infty)$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| \leq \sqrt{|x - y|} < \sqrt{\delta} = \varepsilon.$$

Hence f is uniformly continuous on $[0, \infty)$.

(b) Let $x_n = \sqrt{2n\pi + \frac{\pi}{2}}$ and $y_n = \sqrt{2n\pi}$ for $n \geq 1$. Then

$$|x_n - y_n| = \sqrt{2n\pi + \frac{\pi}{2}} - \sqrt{2n\pi} = \frac{\frac{\pi}{2}}{\sqrt{2n\pi + \frac{\pi}{2}} + \sqrt{2n\pi}} \to 0 \quad \text{as } n \to \infty.$$

However, for all $n \in \mathbb{N}$,

$$|g(x_n) - g(y_n)| = |\cos(2n\pi + \frac{\pi}{2}) - \sin(2n\pi)| = |0 - 1| = 1.$$

By Nonuniform Continuity Criteria, g is not uniformly continuous on $\mathbb{R}$.

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Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $\mathbb{R}$ with $f(0) = 0$. Prove that there exists some $C > 0$ such that

$$|f(x)| \leq 1 + C|x| \qquad \text{for all } x \in \mathbb{R}.$$

Solution. Let $\varepsilon_0 := 1$. Since f is uniformly continuous on $\mathbb{R}$, there exists $\delta > 0$ such that

$$|f(u) - f(v)| < \varepsilon_0 = 1 \qquad \text{whenever } |u - v| < \delta.$$

Let $x \in \mathbb{R}$. Choose $N \in \mathbb{N}$ such that $N - 1 \leq |x|/\delta < N$. Now

$$\begin{aligned}|f(x)| &= |f(x) - f(0)| + |f(0)| \\ &\leq \sum_{n=1}^{N} |f(n \cdot \frac{x}{N}) - f((n-1) \cdot \frac{x}{N})| + 0 \\ &\leq N \\ &= 1 + (N - 1) \\ &\leq 1 + \frac{1}{\delta}|x|.\end{aligned}$$

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Example 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \ell \in \mathbb{R}$. Show that f is uniformly continuous on $\mathbb{R}$.

Example 4. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic on $\mathbb{R}$ if there exists a number $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$.