MATH2050B 2021 HW 5

TA's solutions[1] to selected problems

Q1. Let $\sum_{n=1}^{\infty} a_n$ be a positive series and $\sum_{n=1}^{\infty} 2^n a_{2^n}$ be its condensed one and let

$$s_n := \sum_{i=1}^{n} a_i, \ \ \forall n = 1, 2, \dots$$

$$t_n := \sum_{i=1}^{n} 2^i a_{2^i}, \ \ \forall n = 1, 2, \dots$$

Suppose (a_n) decreases to 0. Show that $\sum_{n=1}^{\infty} a_n < +\infty$ iff $\sum_{n=1}^{\infty} 2^n a_{2^n} < +\infty$ (known as Cauchy Condensation test) along the following steps: show that for all $n \geq 2$:

(i) $s_{2^n-1} \leq a_1 + t_{n-1}$

(ii) $s_{2^n} \geq a_1 + \frac{t_n}{2}$

Solution. To show (i), observe that: $a_2 + a_3 \leq 2a_2$, $a_4 + a_5 + .. + a_7 \leq 4a_4$, etc. Formally, for every $i = 1, 2, \dots$

$$a_{2^i} + a_{2^i+1} + \cdots + a_{2^i+(2^i-1)} \leq 2^i a_{2^i}$$

Therefore

$$\begin{aligned} s_{2^n-1} &= \sum_{i=1}^{2^n-1} a_i = a_1 + \sum_{i=1}^{n-1} \sum_{k=0}^{2^i-1} a_{2^i+k} \\ &\leq a_1 + \sum_{i=1}^{n-1} 2^i a_{2^i} = a_1 + t_{n-1} \end{aligned}$$

For (ii), observe that: $2a_4 \leq a_3 + a_4$, $4a_8 \leq a_5 + a_6 + \cdots + a_8$, etc. Formally, for every $i = 1, 2, \dots$

$$2^i a_{2^{i+1}} \leq a_{2^i+1} + a_{2^i+2} + \cdots + a_{2^i+2^i}$$

Therefore

$$\begin{aligned} a_1 + \frac{t_n}{2} &= a_1 + \sum_{i=1}^{n} 2^{i-1} a_{2^i} \\ &\leq a_1 + a_2 + \sum_{i=1}^{n-1} 2^i a_{2^{i+1}} \\ &\leq a_1 + a_2 + \sum_{i=1}^{n-1} \sum_{k=1}^{2^i} a_{2^i+k} \\ &\leq a_1 + a_2 + \sum_{i=3}^{2^n} a_i \end{aligned}$$

Finally let us prove that (t_n) converges iff (s_n) converges. Because both (t_n) and (s_n) are increasing sequence, so to show convergence it suffices to show boundedness.

[1] please kindly send an email to `nclliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

Assume (s_n) converges, then (s_n) is bounded by some M, then using (ii), we have $t_n \le 2M - 2a_1$ for all n. Hence (t_n) converges. Conversely assume (t_n) converges, note that $s_n \le s_{2^n}$ for all n, so $s_n \le s_{2^n} \le a_1 + t_{n-1}$, which shows (s_n) is bounded.

Q2. For any bounded sequence (x_n):

$$\limsup_n s_n := \lim_n s_n = \max L = \inf E$$

where

- $s_n := \sup\{x_m : n \le m\}$
- $L = \{\ell \in \mathbb{R} : \exists \text{ some subseq of } (x_n) \text{ convergent to } \ell\}$
- $E = \{u \in \mathbb{R} : \exists N \in \mathbb{N} \text{ s.t. } x_n \le u \,\forall n \ge N\}$

Is it true that $\limsup x_n \in E$?

Solution. Not true. Consider the sequence (x_n) defined by $x_n = \frac{1}{n}$. Then (x_n) is convergent to 0, $\limsup x_n = 0$. In this case $E = (0, \infty)$

Q3. Recall that for $x_0 \in \mathbb{R}$ and $\delta > 0$, $V_\delta(x_0) := \{x \in \mathbb{R} : |x - x_0| < \delta\}$. Check all equalities below: (**Remark:** there was a typo in the definition of A^c, corrected in here)

$$\begin{aligned} A^c &:= \{c \in \mathbb{R} : V_\delta(c) \text{ intersects } A \setminus \{c\} \,\forall \delta > 0\} \\ &= \{c \in \mathbb{R} : \forall \delta > 0 \,\exists a \in A \text{ s.t. } 0 < |a - c| < \delta\} \ldots \text{ (call this set } A_1) \\ &= \{c \in \mathbb{R} : \forall n \in \mathbb{N} \,\exists a_n \in A \setminus \{c\} \text{ s.t. } |a_n - c| < \frac{1}{n}\} \ldots (A_2) \\ &= \{c \in \mathbb{R} : \exists \text{ a seq } (a_n) \text{ in } A \setminus \{c\} \text{ s.t. } \lim_n a_n = c\} \ldots (A_3) \\ &= \{c \in \mathbb{R} : \text{dist}(c, A \setminus \{c\}) = 0\} \ldots (A_4) \end{aligned}$$

where $\text{dist}(x, B) = \inf\{|x - b| : b \in B\}$ for all nonempty $B \subset \mathbb{R}$.

Solution. We prove that $A^c \subset A_1 \subset A_2 \subset A_3 \subset A_4 \subset A^c$.

$(A^c \subset A_1)$ Let $c \in A^c$. Let $\delta > 0$. Then $V_\delta(c) \cap A \setminus \{c\}$ is nonempty, pick a point a in this intersection, then $a \in A$, $a \ne c$, and so $0 < |a - c| < \delta$. $c \in A_1$

$(A_1 \subset A_2)$ Let $c \in A_1$. Let $n \in \mathbb{N}$. For $\delta = \frac{1}{n}$, there exists $a \in A$ such that $0 < |a - c| < \frac{1}{n}$. It follows that $a \ne c$ or otherwise $0 = |a - c|$. $c \in A_2$

$(A_2 \subset A_3)$ Let $c \in A_2$. For every $n \in \mathbb{N}$, there exists $a_n \in A \setminus \{c\}$ such that $|a_n - c| < \frac{1}{n}$. Then the sequence (a_n) converges to c. $c \in A_3$.

$(A_3 \subset A_4)$ Let $c \in A_3$. Note that the dist function is non-negative, i.e. $\text{dist}(x, B) \ge 0$. Let (a_n) be a sequence in $A \setminus \{c\}$ convergent to c. Then for all n:

$$0 \le \text{dist}(c, A \setminus \{c\}) \le |c - a_n|$$

Taking $n \to \infty$ we see that $\text{dist}(c, A \setminus \{c\}) = 0$. $x \in A_3$

$(A_4 \subset A^c)$ Let $c \in A_4$. Let $\delta > 0$. Note $\text{dist}(c, A \setminus \{c\}) < \delta$, by definition of infimum there exists $a \in A \setminus \{c\}$ such that $|a - c| < \delta$, i.e. $V_\delta(c) \cap A \setminus \{c\} \ne \varnothing$. $c \in A^c$.

Q4. Let $A := (1, \sqrt{2}) \cap \mathbb{Q}$. Identify A^c with each of the following methods:

(a) Check via definition given in Q3.

(b) Let $f_c(x) = \text{dist}(x, A \setminus \{c\})$ for all $x \in \mathbb{R}$. Determine f_c and hence identify A^c.

Solution. We check that $A^c = [1, \sqrt{2}]$ in each of (a), (b):

(a): ($A^c \subset [1, \sqrt{2}]$). Let $c \in A^c$, suppose on the contrary that $c \in (-\infty, 1)$ or $c \in (\sqrt{2}, \infty)$. For the first case, there exists a $\delta > 0$ such that $V_\delta(c) \subset (-\infty, 1)$. For the second case, there exists a $\delta > 0$ such that $V_\delta(c) \subset (\sqrt{2}, \infty)$. In both cases, there exists $\delta > 0$ with $V_\delta(c) \cap A \setminus \{c\} = \varnothing$.

($[1, \sqrt{2}] \subset A^c$). Let $c \in [1, \sqrt{2}]$. Let $\delta > 0$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, so $V_\delta(c) \cap A \setminus \{c\} \neq \varnothing$. Hence $c \in A^c$.

(b): $f_c(x) = 0$ if $x \in [1, \sqrt{2}]$, $f_c(x) = 1 - x$ if $x < 1$ and $f_c(x) = x - \sqrt{2}$ if $x \geq \sqrt{2}$. Hence $f_c(c) = 0$ iff $c \in [1, \sqrt{2}]$. In **Q3** we proved $A^c = A_4$. Hence $A^c = [1, \sqrt{2}]$.

Q5. Let $x_0 \in A^c$, $f : A \to \mathbb{R}$ and $\ell_1, \ell_2 \in \mathbb{R}$. Suppose $f(x) \to \ell_i$ $(i = 1, 2)$ as $x \to x_0$ $(x \in A)$. Show that $\ell_1 = \ell_2$.

Solution. Let $\epsilon > 0$, then there exists $\delta > 0$ such that for all $x \in A$ with $0 < |x - x_0| < \delta$, $|f(x) - \ell_i| < \epsilon/2$. Then $|\ell_1 - \ell_2| < |f(x) - \ell_1| + |f(x) - \ell_2| < \epsilon$. Hence $\ell_1 = \ell_2$.