Midterm Exam

Q5: Consider the following iterative scheme

$$x^{k+1} = Gx^k + b.$$

$G = QJQ^{-1}$ where $Q \in M_{n\times n}(\mathbb{R})$, $J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{pmatrix}$ is a

block-diagonal matrix, $J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$.

Assume $I-G$ is invertible. Prove that the above iterative scheme converges within n iterations <u>for any</u> initialization if and only if $\lambda_i = 0$ for all $1 \le i \le m$,

Proof: $\quad x^{k+1} = Gx^k + b, \quad x^* = Gx^* + b.$

$$\Rightarrow e^{k+1} = Ge^k = G^2 e^{k-1} = \cdots = G^k e_1.$$

$$= QJ\cancel{Q^{-1}Q}JQ^{-1} \cdots QJQ^{-1}e_1$$

$$= QJ^kQ^{-1}e_1.$$

$$\Rightarrow Q^{-1}e^{k+1} = J^kQ^{-1}e_1$$

Convergence within finite iteration

$\Leftrightarrow$ For any e_1, $\exists\, k_0$, s.t., $J^{k_0}Q^{-1}e_1 = 0$,

$\Leftarrow$ For any $\tilde{e}_1$, $\exists\, k_0$, s.t., $J^{k_0}\tilde{e}_1 = 0$

"$\Rightarrow$" Motivation of this question

(Brief introduction to Jordan canonical form:)

Def: A vector v is called generalized eigenvector of a linear map T w.r.t to the eigenvalue λ of $T \in \mathcal{L}(V)$ if $(T-\lambda I)^k v = 0$ for some k.

The generalized eigenspace $G(\lambda, T)$ of T w.r.t λ is defined as

$$G(\lambda, T) := \{v : (T-\lambda I)^k v = 0 \text{ for some } k\}.$$

Thm: $V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_n, T)$,

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of T.

Remark: if T is invertible, then

$$V = E(\lambda_1, T) \oplus E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_n, T),$$

eigenspace.

Brief idea of Jordan form:

$G(\lambda, T)$: Pick $v \in G(\lambda, T)$, $v \neq 0$.

if $(T-\lambda I)v, (T-\lambda I)^2 v \cdots, (T-\lambda I)^{k-1} v \neq 0$, $(T-\lambda I)^k v = 0$

$A := \text{span}\{v, Tv, \cdots, T^{k-1}v\}, = \text{span}\{T^{k-1}v, \cdots, Tv, v\}$.

$$(T-\lambda I)|_A = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ \vdots & \vdots & 0 & 1 & \\ & & \vdots & 0 & \ddots \\ 0 & 0 & \cdot & & \ddots \end{pmatrix}$$

$$T|_A = (T-\lambda I)|_A + \lambda I|_A$$

$$= \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} + \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

Jordan form of T is a block-diagonal matrix, $\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{pmatrix}$,

such that $J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$.

So, the iterative scheme converges within n iterations

$\Leftrightarrow$ $\forall\, v \in V$, $J^k v = 0$ for some k.

$\Leftrightarrow$ Any $v \in V$ is a generalized eigenvector of J.

Proof of "$\Rightarrow$":

If $\rho(G) \neq 0$, then $\exists\, \lambda_i$ s.t., $\lambda_i \neq 0$.

Choose an eigenvector $v \neq 0$ w.r.t λ_i, as e_1.

Then, $e_k = G^{k-1} e_1 \neq 0 \quad \forall\, k$.

"$\Leftarrow$": By direct computation.

Q1:

$$E(u) = \int_0^1\int_0^1 \alpha(x,y)\left(\frac{\partial u}{\partial x}\right)^2 + 2\beta(x,y)\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \gamma(x,y)\left(\frac{\partial u}{\partial y}\right)^2 dx\,dy.$$

$u(0,y)=0,\ u(1,y)=1,\ u(x,0)=u(x,1)=x,$

(a). $w(x,0)=w(x,1)=w(0,y)=w(1,y)=0.$

$$\left.\frac{d}{dt}\right|_{t=0} E(u+tw)$$

$$= \left.\frac{d}{dt}\right|_{t=0}\int_0^1\int_0^1 \alpha(x,y)\left(\frac{\partial u}{\partial x}+t\frac{\partial w}{\partial x}\right)^2 + 2\beta(x,y)\left(\frac{\partial u}{\partial x}+t\frac{\partial w}{\partial x}\right)\left(\frac{\partial u}{\partial y}+t\frac{\partial w}{\partial y}\right)$$

$$+\gamma(x,y)\left(\frac{\partial u}{\partial y}+t\frac{\partial w}{\partial y}\right)^2 dx\,dy$$

$$= \left.\frac{d}{dt}\right|_{t=0}\int_0^1\int_0^1 \alpha(x,y)\left(\left(\frac{\partial u}{\partial x}\right)^2 + 2t\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + t^2\left(\frac{\partial w}{\partial x}\right)^2\right)$$

$$+2\beta(x,y)\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + t\frac{\partial w}{\partial x}\frac{\partial u}{\partial y} + t\frac{\partial u}{\partial x}\frac{\partial w}{\partial y} + t^2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$+\gamma(x,y)\left(\left(\frac{\partial u}{\partial y}\right)^2 + 2t\frac{\partial u}{\partial y}\frac{\partial w}{\partial y} + t^2\left(\frac{\partial w}{\partial y}\right)^2\right) dx\,dy$$

$$= \int_0^1\int_0^1 \left.\frac{d}{dt}\right|_{t=0}\Bigg[\alpha(x,y)\left(\left(\frac{\partial u}{\partial x}\right)^2 + 2t\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \underbrace{t^2\left(\frac{\partial w}{\partial x}\right)^2}_{2t\left(\frac{\partial w}{\partial x}\right)^2}\right)$$

$$+2\beta(x,y)\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + t\frac{\partial w}{\partial x}\frac{\partial u}{\partial y} + t\frac{\partial u}{\partial x}\frac{\partial w}{\partial y} + t^2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$+\gamma(x,y)\left(\left(\frac{\partial u}{\partial y}\right)^2 + 2t\frac{\partial u}{\partial y}\frac{\partial w}{\partial y} + t^2\left(\frac{\partial w}{\partial y}\right)^2\right)\Bigg] dx\,dy$$

$$= \int_0^1\int_0^1 2\alpha(x,y)\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + 2\beta(x,y)\frac{\partial w}{\partial x}\frac{\partial u}{\partial y} + 2\beta(x,y)\frac{\partial u}{\partial x}\frac{\partial w}{\partial y} + 2\gamma(x,y)\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\,dx\,dy.$$

(b) If you are familiar with divergence theorem/integration by part,

$$\frac{d}{dt}\Big|_{t=0} E(u+tw) = 2\int_0^1\int_0^1 \left(\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\nabla u\right)\cdot \nabla w \, dx\, dy$$

$$= \underbrace{2\int_{\partial[0,1]^2} \left(\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\nabla u\cdot \vec{n}\right) w \, dS_y}_{=0} - 2\int_{[0,1]^2} \nabla\cdot\left(\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\nabla u\right) w\, dy$$

$$= -2\int_{[0,1]^2} \nabla\cdot\left(\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\nabla u\right) w\, dy$$

$$= 0 \quad \text{for any } w$$

$$\Rightarrow \nabla\cdot\left(\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\nabla u\right) = 0$$

Another method:

Actually we can use a method similar to the 1D case, since the domain $[0,1]\times[0,1]$ is so nice.

$$\frac{d}{dt}\Big|_{t=0} \frac{1}{2}E(u+tw) = \int_0^1\int_0^1 \alpha(x,y)\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\, dx\, dy + \int_0^1\int_0^1 \beta(x,y)\frac{\partial u}{\partial y}\frac{\partial w}{\partial x}\, dx\, dy$$

$$+\int_0^1\int_0^1 \beta(x,y)\frac{\partial u}{\partial x}\frac{\partial w}{\partial y}\, dx\, dy + \int_0^1\int_0^1 \gamma(x,y)\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\, dx\, dy$$

$$= \int_0^1\int_0^1 \alpha(x,y)\frac{\partial u}{\partial x}\, dw\, dy + \int_0^1\int_0^1 \beta(x,y)\frac{\partial u}{\partial y}\, dw\, dy$$

$$+\int_0^1\int_0^1 \beta(x,y)\frac{\partial u}{\partial x}\, dw\, dx + \int_0^1\int_0^1 \gamma(x,y)\frac{\partial u}{\partial y}\, dw\, dx$$

$$= \int_0^1\left(\underbrace{\alpha(x,y)\frac{\partial u}{\partial x} w\Big|_0^1}_{=0} - \int_0^1 w\frac{\partial}{\partial x}\left(\alpha(x,y)\frac{\partial u}{\partial x}\right) dx\right) dy$$

$$+\int_0^1 \Big(\underbrace{\beta(x,y)\frac{\partial u}{\partial y} w \Big|_0^1}_{=0} - \int_0^1 w \frac{\partial}{\partial x}\Big(\beta(x,y)\frac{\partial u}{\partial y}\Big)\,dx \Big)\,dy$$

$$+\int_0^1 \Big(\underbrace{\int_0^1 \beta(x,y)\frac{\partial u}{\partial x} w \Big|_0^1}_{=0} - \int_0^1 w \frac{\partial}{\partial y}\Big(\beta(x,y)\frac{\partial u}{\partial x}\Big)\,dy \Big)\,dx$$

$$+\int_0^1 \Big(\underbrace{\int_0^1 \gamma(x,y)\frac{\partial u}{\partial y} w \Big|_0^1}_{=0} - \int_0^1 w \frac{\partial}{\partial y}\Big(\gamma(x,y)\frac{\partial u}{\partial y}\Big)\,dy \Big)\,dx$$

$$= -\int_0^1\int_0^1 w \Big(\frac{\partial}{\partial x}\Big(\beta(x,y)\frac{\partial u}{\partial y} + \alpha(x,y)\frac{\partial u}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\beta(x,y)\frac{\partial u}{\partial x} + \gamma(x,y)\frac{\partial u}{\partial y}\Big)\Big)\,dx\,dy$$

$$= -\int_0^1\int_0^1 w\, \nabla\cdot\left(\begin{pmatrix}\alpha & \beta \\ \beta & \gamma\end{pmatrix}\nabla u\right)$$

Hence, $\nabla\cdot\left(\begin{pmatrix}\alpha & \beta \\ \beta & \gamma\end{pmatrix}\nabla u\right) = 0$