How to accelerate an iterative scheme?

$$\vec{X}_{k+1} = M\vec{x}_k + \vec{f}.$$

Observation: the convergence rate is controlled by $\rho(M)$.

$$\begin{aligned} \vec{X}_{k+1} &= M\vec{x}_k + \vec{f} \\ \vec{X}^* &= M\vec{x}^* + f \end{aligned} \Rightarrow \quad e_{k+1} = Me_k = M^2 e_{k-1} = \cdots = M^k e_1.$$

$$e_1 = x_1 - x^*.$$

Suppose the matrix M has eigenvalues $\lambda_1, \cdots, \lambda_n$, s.t., $\rho(M) < 1$, with corresponding eigenvectors $v_1, \cdots, v_n$.

Let $e_1 = a_1 v_1 + \cdots + a_n v_n$.

If further suppose $1 > |\lambda_1| > |\lambda_2| > \cdots \geq 0$,

then, $M^k e_1 = M^k(a_1 v_1 + \cdots + a_n v_n)$

$$= a_1 \lambda_1^k v_1 + \cdots + a_n \lambda_n^k v_n.$$

$\Rightarrow$ the convergence rate is determined by $\rho(M)$.

Example: (Accelerate using a shift).

$X_{k+1} = (I - tA)x_k + tb$, A is symmetric positive definite.

Also, A has eigenvalues $\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 > 0$.

(a) Show that convergence $\Leftrightarrow$ $0 < t < \frac{2}{\lambda_n}$

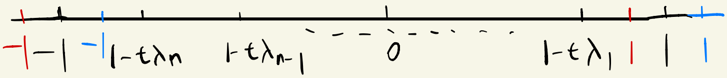
(b) $t_{opt} = \frac{2}{\lambda_1 + \lambda_n}$.

(a). Note that the eigenvalues of $I - tA$ are

$$1-t\lambda_1 \geq 1-t\lambda_2 \geq \cdots \geq 1-t\lambda_n .$$

So, $\rho(I-tA)<1 \iff \begin{cases} 1-t\lambda_1 < 1 \\ 1-t\lambda_n > -1 \end{cases} \iff 0<t<\frac{2}{\lambda_n}.$

(b)



$$-(1-t\lambda_n) = 1-t\lambda_1 \Rightarrow t=\frac{2}{\lambda_1+\lambda_n}$$

SOR method:

$$\left(L+\frac{1}{\omega}D\right)\vec{x}^{k+1} = \left(\frac{1}{\omega}D-(D+U)\right)\vec{x}^{k} + \vec{b}.$$

Example: Consider the linear system $Ax=b$.

$$A=\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad M=\begin{pmatrix} \frac{2}{\omega} & 0 \\ -1 & \frac{2}{\omega} \end{pmatrix} = L+\frac{1}{\omega}D$$

$$B=M^{-1}(M-A)=\begin{pmatrix} 1-\omega & \frac{1}{2\omega} \\ \frac{\omega(1-\omega)}{2} & \frac{\omega^2}{4}-\omega+1 \end{pmatrix}$$

$$f(\lambda)=\det(B-\lambda I) = \lambda^2-\left(\frac{\omega^2}{4}-2\omega+2\right)\lambda+\frac{\omega^2}{4}\left(\frac{2}{\omega}-2\right)^2$$

$$=\left(\lambda-\left(\frac{\omega^2}{8}-\omega+1\right)\right)^2+\frac{\omega^2}{4}\left(\frac{2}{\omega}-2\right)^2-\left(\frac{\omega^2}{8}-\omega+1\right)^2$$

$$\frac{\omega^2}{4}\left(\frac{2}{\omega}-2\right)^2-\left(\frac{\omega^2}{8}-\omega+1\right)^2=0$$

$$\iff \omega=0,\ 8-4\sqrt{2},\ 4(2+\sqrt{3})$$

Take $w = 8 - 4\sqrt{3}$,

then $\rho(B) = \frac{w^2}{8} - w + 1 \approx 0.07$.

We can expect $e_{k+1} = 0.07\, e_k$.

Jacobi method : $\rho \approx 0.5$

G-S method : $\rho \approx 0.25$.

Theorem : (D. Young)

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho^2}}$$ under some conditions,

$$\rho(M_{SOR}, \omega_{opt}) = \omega_{opt} - 1.$$

Q & A session about the midterm exam !

Q6 in HW3:

$a = (a_0, a_1, \cdots, a_{N-1})$, $b = (b_0, b_1, \cdots, b_{N-1})$, $N = 2^s$.

$(b_1, b_2, \cdots, b_{N-1}, b_0)$

$(b_2, b_3, \cdots, b_0, b_1)$

$\vdots$

$c = a * \hat{b}$

How can we apply fft to solve for c?