1 Mathematical modeling through differential equations

In our real world, we see many physical phenomena every day. All these phenomena always follow some rules, which are called physical laws. The mathematical modeling is to formulate a physical phenomenon in terms of mathematical equations using some known physical laws. Then one needs only to solve or analyse the mathematical model equations in order to understand a physical phenomenon.

The mathematical equations of primary interest to this course are called *differential equations*, which are equations involving some unknown functions and their derivatives.

1.1 Modeling an elastic bar

Consider a continuous elastic bar[1] of length 1, which is hanged vertically. (it is displaced up and down due to gravity). Set up an x-axis along the bar, so that its positive direction pointing downwards and its origin is located at the top of the elastic bar. Consider any point at x along the bar (the position is at x if no external force present), it is displaced down to $x + u(x)$ because of the action of the external force of gravity[2]. Function $u(x)$ is called the displacement. The stretching at any point is measured by the derivative $e = du/dx$, called the *strain*. If u is a constant, the elastic bar is unstretched. Otherwise the stretching of the bar produces an internal force (one can experience this force easily by pulling the two ends of an elastic bar). By experiments, people find this internal force is proportional to the strain in the bar, i.e.

$$\text{(internal force)} \quad w(x) = c(x)\,\frac{du}{dx}\ ,$$

where $c(x)$ is a constant determined by the elastic material, or a function if the material is inhomogeneous.

To set up the model, we take a small piece of the bar $[x, x + \triangle x]$, its equilibrium requires all forces acted on it to be balanced. We have

$$\left(c(x)\frac{du}{dx}\right)_{x+\triangle x} - \left(c(x)\frac{du}{dx}\right)_{x} + (\rho\triangle x a)g = 0, \tag{1.1}$$

where g is the gravitational constant, a the cross-sectional area, and $\rho(x)$ the density at position x.

[1]You may pull back and forth an elastic bar and its length is much bigger than its size of cross-section.

[2]Some other external force may be considered.

Dividing both sides of equation (1.1) by $\triangle x$, we get

$$-\frac{d}{dx}(c(x)\ \frac{du}{dx}) = f(x) \tag{1.2}$$

where $f(x) = ag\,\rho(x)$, external force per unit length.

The equation (1.2) must come with appropriate physical boundary conditions to ensure it is well-posed.

1.2 Boundary conditions

(a) Both ends of the elastic bar are fixed, so no displacements:

$$u(0) = 0, \quad u(1) = 0\ .$$

This is called Dirichlet boundary conditions.

(b) Top end of the elastic bar is fixed (no displacement), the other end is free (no internal force since it is in the air):

$$u(0) = 0, \quad w\big|_{x=1} = c(x)\frac{du}{dx}\big|_{x=1} = 0\ .$$

The first is called a Dirichlet boundary condition, the second is called a Neumann boundary condition.

So the complete model for an elastic bar is :

$$-\frac{d}{dx}(c(x)\frac{du}{dx}) = f(x), \quad 0 < x < 1$$

with boundary conditions

$$u(0) = 0, \quad u(1) = 0$$

or

$$u(0) = 0, \quad c(x)\frac{du}{dx}\big|_{x=1} = 0\ .$$

This differential equation is called a two-point boundary value problem[3].

[3]Think about why we need two boundary conditions.

1.3 Solutions of the elastic bar model

We now try to find the solution of the following boundary value problem

$$\begin{cases} -\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) = f(x), \quad 0 < x < 1 \\ u(0) = 0, \quad c(x)\frac{du}{dx}\big|_{x=1} = 0\,. \end{cases} \tag{1.3}$$

Solution. Integrating the equation (1.3) over $(x, 1)$, we obtain

$$-c(x)\frac{du}{dx}\Big|_x^1 = \int_x^1 f(t)\,dt\ ,$$

using the boundary conditions, we have

$$c(x)u'(x) = \int_x^1 f(t)dt\ ,$$

or

$$u'(x) = \frac{1}{c(x)}\int_x^1 f(t)dt\,.$$

Integrating over $(0, x)$ gives

$$u(x) = \int_0^x \frac{1}{c(x)}\int_x^1 f(t)dtdx, \tag{1.4}$$

this is the required exact solution of the problem (1.3). ♯

Example 1.1. *Find the exact solution of the problem*

$$\begin{cases} -\frac{d^2u}{dx^2} = x^2, \quad 0 < x < 1 \\ u(0) = 0, \quad \frac{du}{dx}\big|_{x=1} = 0\,. \end{cases} \tag{1.5}$$

Solution. Integrating the equation (1.5) over $(x, 1)$, we obtain

$$-\frac{du}{dx}\Big|_x^1 = \int_x^1 t^2\,dt\ ,$$

using the boundary conditions, we have

$$u'(x) = \int_x^1 t^2 dt = \frac{1}{3} - \frac{1}{3}x^3.$$

Integrating over $(0, x)$ gives

$$u(x) = \int_0^x (\frac{1}{3} - \frac{1}{3}t^3)dt = \frac{1}{3}x - \frac{1}{12}x^4\,. \tag{1.6}$$

It is easy to verify that this $u(x)$ is really the solution of the system (1.5). ♯

Example 1.2. *Find the exact solution of the following problem*

$$-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) = f(x), \quad 0 < x < 1$$

with the boundary conditions

$$u(0) = -1,\ u(1) = 1\,.$$

<u>**Solution**</u>. Write the equation as

$$-(c(x)u'(x))' = f(x),$$

then integrating over $(x, 1)$, we get

$$c(x)u'(x) = C_0 - \int_x^1 f(t)\,dt,$$

where C_0 is an integration constant. This implies

$$u'(x) = \frac{C_0}{c(x)} - \frac{1}{c(x)}\int_x^1 f(t)\,dt\,.$$

Now integrating over $(0, x)$ gives

$$u(x) = -1 + C_0\int_0^x \frac{1}{c(t)}dt - \int_0^x \frac{1}{c(x)}\int_x^1 f(t)\,dt\,.$$

Using the boundary condition $u(1) = 1$, we can find the integration constant C_0. $\sharp$

1.4 Homogeneous and non-homogeneous boundary conditions

The boundary conditions

$$u(0) = 0, \quad \text{or} \quad u(1) = 0$$

or

$$c(x)\frac{du}{dx}\Big|_{x=1} = 0$$

are all called homogeneous boundary conditions, while the boundary conditions

$$u(0) = 1, \quad \text{or} \quad u(1) = -2,$$

or

$$c(x)\frac{du}{dx}\Big|_{x=1} = -3$$

are all called non-homogeneous boundary conditions.

1.5 Heat conduction model

Consider a bar of length 1. When the bar is heated its temperature will change as time t varies. Assume the heat source can be described by a strength function $f(x,t)$ at position x and time t. Then the temperature $u(x,t)$ of the bar at position x and at time t can be modeled by

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\big(c(x)\frac{\partial u}{\partial x}\big) = f(x,t), \quad 0 < x < 1,\ t \geq 0 \tag{1.7}$$

This is called *the heat conduction equation.* The coefficient function $c(x)$ is the heat conductivity of the material, which can be measured physically.

To complete the mathematical model, we have to impose some initial condition and boundary conditions. These conditions can be imposed according to the physical setting of the system. For example, if the initial temperature of the bar is given, say by $u_0(x)$, then the initial condition can be stated as

$$u(x,0) = u_0(x), \quad 0 \leq x \leq 1\,. \tag{1.8}$$

If one end of the bar, say $x = 0$, is placed at an ice basin, then the boundary condition at $x = 0$ is

$$u(0,t) = 0, \quad t \geq 0; \tag{1.9}$$

if the other end is insulated, then there is no heat flux coming through the end, and the boundary condition at $x = 1$ should be

$$-c(x)\frac{\partial u}{\partial x}\Big|_{x=1} = 0\ . \tag{1.10}$$

The system (1.7)-(1.10) is called an initial-boundary value problem.

1.6 Sturm-Liouville problems

With a little bit generalization of the preceding elastic bar model, we have the following Sturm-Liouville problem

$$-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) + q(x)u = f(x), \quad 0 < x < 1 \tag{1.11}$$

The Sturm-Liouville system, with homogeneous or non-homogeneous boundary conditions may have many physical applications:

(a) In the quantum theory, the equation is called the Schrödinger's equation.

(b) For modeling the oscillations of a drum, it is called the Bessel's equation.

1.6.1 Solutions of Sturm-Liouville problems

In general, it is difficult to find the analytical solutions for the Sturm-Liouville problem (1.11). Only for some special simple coefficients $c(x)$, $q(x)$ and $f(x)$, one may solve the problem (1.11).

Solutions with special coefficients

Let us consider the special case where $c(x) = c$, $q(x) = q$, $f(x) = 0$ and q and c have the same sign, then the equation becomes

$$-c\,u'' + q\,u = 0\ . \tag{1.12}$$

To solve this equation, we multiply its both sides by u' to obtain

$$c\,u''u' = q\,uu'\ ,$$

this can be written as

$$c\{(u')^2\}' = q\,(u^2)'\ ,$$

or

$$c\,(u')^2 = q\,u^2\ ,$$

taking the square-root on both sides gives

$$u' = \pm\sqrt{\frac{q}{c}}\,u\ .$$

By integration, we obtain

$$(\ln u)' = \pm\sqrt{\frac{q}{c}} \ ,$$

or

$$\ln u(x) = \pm\sqrt{\frac{q}{c}}x \ ,$$

so we find two solutions for the equation (1.12):

$$u(x) = \exp\Big(\pm\sqrt{\frac{q}{c}}x\Big) \ .$$

For the general solutions of the system (1.12), we have the following useful result[4]:

> **Any linear combination of**
>
> $$u_1 = \exp\Big(\sqrt{\frac{q}{c}}x\Big), \quad u_2 = \exp\Big(-\sqrt{\frac{q}{c}}x\Big),$$
>
> **i.e.,** $u = \alpha_1 u_1 + \alpha_2 u_2$ **for arbitrary real numbers** α_1 **and** α_2**, is also a solution of the equation (1.12).**

This result tells us that the equation (1.12) has infinitely many solutions. But to determine a specific solution, one needs to impose boundary conditions. For example, if we have the Dirichlet boundary conditions

$$u(0) = 0, \quad u(1) = 2 \ ,$$

then we can determine a specific solution:

$$u(x) = \alpha_1 u_1(x) + \alpha_2 u_2(x) = \frac{2}{e^{\sqrt{q/c}} - e^{-\sqrt{q/c}}}\ (e^{\sqrt{q/c}x} - e^{-\sqrt{q/c}x}) \ .$$

Example. Determine the solution of (1.12), which satisfies the boundary conditions:

$$u'(0) = 0, \quad u(1) = 2 \, .$$

Solutions for non-homogeneous case

If we add a source term $f(x)$ to the right-hand side of (1.12), the Sturm-Liouville equation becomes

$$-c\,u'' + q\,u = f(x), \quad 0 < x < 1 \, . \tag{1.13}$$

[4]Please check this conclusion yourself.

How to find the solution of this equation ?

It is not so simple for general function $f(x)$. But if we know a special solution $w(x)$ of (1.13), then we can easily check that the combination

$$u(x) = \alpha_1 u_1(x) + \alpha_2 u_2(x) + w(x)$$

is a solution of (1.13) for any real numbers α_1 and α_2.

Now, using this fact, find the solution of the following Sturm-Liouville equation

$$\begin{cases} -c\,u'' + q\,u = 1 + x\,, \quad 0 < x < 1 \\ u'(0) = 0, \quad u(1) = 2\,. \end{cases} \tag{1.14}$$

It is easy to see that $w(x) = (1 + x)/q$ is a special solution to the equation, so the combination

$$u(x) = \alpha_1 u_1(x) + \alpha_2 u_2(x) + (1 + x)/q$$

is a solution of (1.13) for any real numbers α_1 and α_2. The two coefficients α_1 and α_2 can be determined by the boundary conditions in (1.14).

• Think about whether we can apply the previous technique to find the solution of the following Sturm-Liouville equation

$$\begin{cases} -c\,u'' + q\,u = f(x)\,, \quad 0 < x < 1 \\ u'(0) = 0, \quad u(1) = 2\,. \end{cases}$$

Is it possible to find a solution of trigonometric function or quadratic polynomial when $f(x) = \sin x$ or x^2 ?

1.6.2 Sturm-Liouville operator

For our later use, we introduce the following inner product

$$(u, v) = \int_0^1 u(x)v(x)dx \quad \forall\, u, v \in L^2(0, 1)$$

where the space $L^2(0, 1)$ is given by

$$L^2(0, 1) = \left\{v; \quad \int_0^1 v^2(x)dx < \infty\right\}\,.$$

We will frequently use the following formula of integration by parts:

$$\int_0^1 \frac{du}{dx} v dx = -\int_0^1 u \frac{dv}{dx} dx + [uv]\Big|_{x=0}^{x=1}\,.$$

Let $C^2(0,1)$ be the space with all second order continuously differentiable functions in $(0,1)$. We now define the following operator

$$Au = -\frac{d}{dx}(c(x)\frac{du}{dx}) + q(x)u \quad \forall\, u \in C^2(0,1).$$

This operator A is called the Sturm-Liouville operator.

For the Sturm-Liouville operator, we have for any $u, v \in C^2(0,1)$,

$$\begin{aligned}(Au, v) &= \int_0^1 \{-\frac{d}{dx}(c(x)\frac{du}{dx})v + q(x)uv\}dx \\ &= \int_0^1 (c(x)\frac{du}{dx}\frac{dv}{dx} + q(x)uv)dx - (c(x)\frac{du}{dx}v)\Big|_{x=0}^{x=1}. \end{aligned} \tag{1.15}$$

This is a very useful relation.

1.7 Variational formulations for differential equations

Recall the model equation for an elastic bar,

$$-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) + q(x)u(x) = f(x),\ 0 < x < 1 \tag{1.16}$$

with boundary conditions

$$u(0) = 0, \qquad c(x)\frac{du}{dx}\Big|_{x=1} = 0 \tag{1.17}$$

One important method to study the properties of the solutions to the equations (1.16)-(1.17) is to use the integral form, often called the variational formulation.

Next, we shall discuss how to derive the variational formulation for the differential equation (1.16)-(1.17). The same methodology can be applied to any other second order differential equations.

The derivation is standard and simple. To do so, we multiply both sides of equation (1.16) by an arbitrary test function v satisfying $v(0) = 0$ to obtain

$$-\frac{d}{dx}(c(x)\frac{du}{dx})v + q(x)uv = f(x)v\ ,$$

then integrating over $(0,1)$ gives

$$\int_0^1 \left(-\frac{d}{dx}(c(x)\frac{du}{dx})v + q(x)uv\right) dx = \int_0^1 f(x)v dx\ . \tag{1.18}$$

Now by integration by parts and the boundary conditions (1.17), we have

$$\int_0^1 \left(c(x)\frac{du}{dx}\frac{dv}{dx} + q(x)uv \right) dx = \int_0^1 f(x)v\ dx.$$

This leads to the **variational formulation** for the equations (1.16)-(1.17):

Find the solution u such that $u(0) = 0$ and

$$a(u,v) = g(v) \quad \text{for any } v \text{ satisfying } v(0) = 0 \tag{1.19}$$

where $a(\cdot,\cdot)$ and $g(\cdot)$ are given by

$$\begin{aligned} a(u,v) &= \int_0^1 \left(c(x)\frac{du}{dx}\frac{dv}{dx} + q(x)uv \right) dx\,, \\ g(v) &= \int_0^1 f(x)v\ dx\,. \end{aligned}$$

One can check that $a(\cdot,\cdot)$ is linear with respect to each variable, and is symmetric, i.e., for any u and v,

$$a(u,v) = a(v,u)\,.$$

Furthermore, we know that $a(\cdot,\cdot)$ is also positive, i.e.,

$$a(v,v) > 0 \quad \forall\, v \neq 0\,.$$

Equivalence between boundary value and variational problems

In the following, we shall verify that

The boundary value problem (1.16)-(1.17) is equivalent to the variational problem (1.19).

First, we know already that the solution u of the boundary value problem (1.16)-(1.17) is also a solution to the variational equation (1.19). Next, we will confirm that any solution u of (1.19) is also a solution of the boundary value problem (1.16)-(1.17).

In fact, since u satisfies (1.19), we have

$$\int_0^1 \left(c(x)\frac{du}{dx}\frac{dv}{dx} + q(x)uv \right) dx = \int_0^1 f(x)v\ dx \quad \forall\ v \text{ with } v(0) = 0\ .$$

Using integration by parts, we obtain

$$\int_0^1 \left(-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big)v + q(x)uv \right)\ dx + c(x)\frac{du}{dx}v\Big|_{x=0}^{x=1} = \int_0^1 f(x)v\ dx\ . \tag{1.20}$$

As the test function v is arbitrary, we can take v to be arbitrary but satisfying the boundary conditions $v(0) = v(1) = 0$, then (1.20) becomes

$$\int_0^1 \Big\{ -\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) + q(x)u - f \Big\} v \, dx = 0 \quad \text{for any } v \text{ with } v(0) = v(1) = 0 \ ,$$

this implies

$$-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) + q(x)u = f, \quad 0 < x < 1. \tag{1.21}$$

Substituting this into (1.20), we have

$$c(1)u_x(1)v(1) = 0 \quad \text{for any} \ \ v \text{ with } v(0) = 0 \,,$$

this indicates that u also satisfies the condition

$$c(x)\frac{du}{dx}\Big|_{x=1} = 0 \ . \tag{1.22}$$

(1.21) and (1.22) tell us that u is a solution of the boundary value problem (1.16)-(1.17). ♯

Equivalence between boundary value and minimization problems

Now we investigate the relation between the boundary value problem (1.16)-(1.17) and the following potential energy functional

$$J(u) = \frac{1}{2}\int_0^1 \Big(c(x)\big(\frac{du}{dx}\big)^2 + q(x)u^2 \Big) \, dx - \int_0^1 f(x)\, u(x)\, dx,$$

we are going to verify the following relations:

The function u that minimizes $J(v)$ over all v satisfying $v(0) = 0$ must be the solution of the system (1.16)-(1.17), that is, it satisfies the differential equation

$$-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) + q(x)u = f(x), \quad 0 < x < 1$$

with the boundary conditions

$$u(0) = 0 \quad \textbf{and} \quad c\frac{du}{dx}\Big|_{x=1} = 0 \ .$$

The converse is also true.

To see this, let u minimize $J(u)$, so we have

$$J(u) \leq J(v) \quad \forall\, v \text{ with } \quad v(0) = 0\ . \tag{1.23}$$

Consider a real function

$$F(t) = J(u + tv)\ .$$

Using (1.23) we know

$$F(0) \leq F(t) \quad \forall\, t \in \mathbb{R}^1\ ,$$

that is, $t = 0$ is a minimizer of $F(t)$. This implies

$$F'(0) = 0\ . \tag{1.24}$$

Now by definition,

$$\begin{aligned}
F(t) - F(0) =& J(u + tv) - J(u) \\
=& \frac{1}{2}\Big\{ \int_0^1 \left(c(x)(u_x + tv_x)^2 + q(x)(u + tv)^2 \right) dx - \int_0^1 f(u + tv) dx \Big\} \\
& - \frac{1}{2}\Big\{ \int_0^1 \left(c(x)u_x^2 + q(x)u^2 \right) dx - \int_0^1 fu\ dx \Big\} \\
=& t\Big\{ \int_0^1 (c(x)u_x v_x + q(x)uv) dx - \int_0^1 fv\ dx \Big\} + \frac{1}{2}t^2 \int_0^1 c(x)v_x^2 dx\ ,
\end{aligned}$$

which gives

$$F'(0) = \int_0^1 (c(x)u_x v_x + q(x)uv) dx - \int_0^1 fv dx\ .$$

This with (1.24) yields

$$\int_0^1 c(x) \left(\frac{du}{dx}\frac{dv}{dx} + q(x)uv \right) dx = \int_0^1 fv dx \quad \text{for any } v \text{ with } v(0) = 0\ ,$$

namely, u is a solution of the variational problem (1.19), so it is also a solution of the boundary value problem (1.16)-(1.17).

To see the converse part, for any v such that $v(0) = 0$ we can calculate

$$\begin{aligned}
J(v) - J(u) \quad = \quad & \Big\{ \frac{1}{2}(c\, v_x, v_x) + (q\, v, v) - (f, v) \Big\} \\
& - \Big\{ \frac{1}{2}(c\, u_x, u_x) + (q\, u, u) - (f, u) \Big\} \\
= \quad & \Big\{ \frac{1}{2}\Big(c\,(v - u)_x, (v - u)_x \Big) + (q\,(v - u), v - u) \Big\} \\
& + \Big\{ \Big(c\, u_x, (v - u)_x \Big) + (q\, u, v - u) - (f, v - u) \Big\}\ .
\end{aligned}$$

Using this relation and the equivalence between the boundary value problem (1.16)-(1.17) and the variational problem (1.19), one can easily see that if u is a solution to the boundary value problem (1.16)-(1.17), then it must a minimizer of $J(v)$. $\sharp$

1.8 Further discussions on variational formulations

We now consider a bit more general boundary condition problem:

$$-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) + q(x)u(x) = f(x), \ a < x < b \tag{1.25}$$

with boundary conditions

$$c(x)\frac{du}{dx}\Big|_{x=a} = \alpha, \qquad u(b) = \beta \tag{1.26}$$

Same as we did in the last subsection, we can derive the variational formulation for the system (1.25)-(1.26).

To do so, we multiply both sides of equation (1.25) by an arbitrary test function v satisfying $v(b) = 0$, then integrate over (a, b) to obtain

$$\int_a^b \left(-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big)v + q(x)uv\right) dx = \int_a^b f(x)v dx \ . \tag{1.27}$$

Now using integration by parts and the boundary conditions (1.26), we deduce

$$\int_a^b \left(c(x)\frac{du}{dx}\frac{dv}{dx} + q(x)uv\right) dx = \int_a^b f(x)v \ dx - \alpha\, v(a).$$

This leads to the **variational formulation** for the equations (1.25)-(1.26):

Find the solution u such that $u(b) = \beta$ and

$$a(u, v) = g(v) \quad \text{ for any } v \text{ satisfying } v(b) = 0 \tag{1.28}$$

where $a(\cdot, \cdot)$ and $g(\cdot$ are given by

$$\begin{aligned} a(u, v) &= \int_a^b \left(c(x)\frac{du}{dx}\frac{dv}{dx} + q(x)uv\right) dx\,, \\ g(v) &= \int_a^b f(x)v \ dx - \alpha\, v(a)\,. \end{aligned}$$

Equivalence between boundary value and variational problems

The same as we did in the last subsection, we can verify that

The boundary value problem (1.25)-(1.26) is equivalent to the variational problem (1.28).

First, we know already by the derivation of the variational problem (1.28) that the solution u of the boundary value problem (1.25)-(1.26) is also a solution to the variational equation (1.28). Next, we will confirm that any solution u of (1.28) is also a solution of the boundary value problem (1.25)-(1.26).

In fact, since u satisfies (1.28), we obtain by using integration by parts that for any v satisfying $v(b) = 0$,

$$\int_a^b \Big(-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big)v + q(x)uv \Big)\; dx + c(x)\frac{du}{dx}v\big|_{x=a}^{x=b} = \int_a^b f(x)v\; dx - \alpha\, v(a)\; . \quad (1.29)$$

Now taking all the test functions v which satisfy the boundary conditions $v(a) = v(b) = 0$, then (1.29) becomes

$$\int_a^b \Big\{ -\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) + q(x)u - f\Big\}v\; dx = 0 \qquad \text{for any } v \text{ with } v(a) = v(b) = 0\;,$$

this implies

$$-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) + q(x)u = f, \quad a < x < b. \quad (1.30)$$

Substituting this into (1.29), we have

$$-c(a)u_x(a)v(a) = -\alpha\, v(a) \quad \text{for any} \;\; v \text{ with } v(b) = 0\,,$$

this indicates that u also satisfies the condition

$$c(x)\frac{du}{dx}\big|_{x=a} = \alpha\; . \quad (1.31)$$

(1.30) and (1.31) tell us that u is a solution of the boundary value problem (1.25)-(1.26). ♯

Equivalence between boundary value and minimization problem

Now we investigate the relation between the boundary value problem (1.25)-(1.26) and the following potential energy functional

$$J(u) = \frac{1}{2}\int_b^a \Big(c(x)\big(\frac{du}{dx}\big)^2 + q(x)u^2 \Big)\, dx - \Big\{ \int_a^b f(x)\, u(x)\, dx - \alpha\, v(a)\Big\},$$

we are going to verify the following relations:

The function u that minimizes $J(v)$ over all v satisfying $v(b) = \beta$ must be the solution of the system (1.25)-(1.26), that is, it satisfies the differential equation

$$-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) + q(x)u = f(x), \quad a < x < b$$

with the boundary conditions

$$c\frac{du}{dx}\big|_{x=a} = \alpha\,, \quad u(b) = \beta\,.$$

The converse is also true.

The proof of this equivalence is basically the same as we did in the last subsection. So we omit it here.

1.9 Complementary minimum principle for the internal force

We know from the previous discussions that the displacement u of an elastic bar satisfies the boundary value problem:

$$-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) = f(x)\,, \quad 0 < x < 1 \tag{1.32}$$

and the boundary conditions

$$u(0) = 0, \quad c(x)\frac{du}{dx}\big|_{x=1} = 0\ .$$

Moreover, u also solves the equivalent variational problem

$$\int_0^1 c(x)\frac{du}{dx}\frac{dv}{dx}dx = \int_0^1 f(x)v\ dx \quad \forall\ v \text{ with } v(0) = 0$$

and minimizes the potential energy functional

$$J(u) = \frac{1}{2}\int_0^1 c(x)\Big(\frac{du}{dx}\Big)^2 dx - \int_0^1 f(x)u\ dx\ .$$

Below, we shall discuss some similar results for the internal force $w(x) = c(x)\frac{du}{dx}$. From (1.32) we know w satisfies

$$-\frac{dw}{dx} = f(x), \quad 0 < x < 1 \tag{1.33}$$

and the boundary condition

$$w(1) = 0\,. \tag{1.34}$$

Corresponding to the problem (1.33)-(1.34), we define a new energy functional

$$Q(w) = \frac{1}{2}\int_0^1 \frac{1}{c(x)} w^2(x)dx \quad \Big(\text{recall } w = c(x)\frac{du}{dx}\Big).$$

Consider the minimization problem

$$\min Q(w) \text{ with } w \text{ such that } -\frac{dw}{dx} = f(x), \quad w(1) = 0 \tag{1.35}$$

This is a constrained optimization problem.

To transform the constrained problem into a unconstrained problem, we introduce a *Lagrangian functional*

$$\begin{aligned} L(u,w) &= Q(w) + \int_0^1 u\big(\frac{dw}{dx} + f\big)dx \\ &= \frac{1}{2}\int_0^1 \frac{1}{c(x)} w^2(x)dx + \int_0^1 u\big(\frac{dw}{dx} + f\big)dx\ , \end{aligned}$$

where u is called a *Lagrangian multiplier.*

Now we are going to show that if w^* and u^* are functions such that $w^*(1) = 0$, $u^*(0) = 0$, and are the minimizer and the maximizer of the following problems:

$$L(u^*, w^*) = \min_w L(u^*, w)\,, \quad L(u^*, w^*) = \max_u L(u, w^*)\,, \tag{1.36}$$

then w^* and u^* satisfy

$$w^* = c(x)u^*_x\,, \quad -w^*_x = f\,. \tag{1.37}$$

And the converse is also true.

We first show that if w^* and u^* are functions such that $w^*(1) = 0$, $u^*(0) = 0$, and are the solutions to (1.37), then they are also the solutions to the optimization problems in (1.36). In fact, for any v, we have

$$\begin{aligned} L(u^*, v) - L(u^*, w^*) &= \frac{1}{2}(c^{-1}v, v) + (u^*, v_x + f) - \frac{1}{2}(c^{-1}w^*, w^*) - (u^*, w^*_x + f) \\ &= \frac{1}{2}(c^{-1}(v - w^*), v - w^*) + (c^{-1}(v - w^*), w^*) + (u^*, (v - w^*)_x) \\ &= \frac{1}{2}(c^{-1}(v - w^*), v - w^*) + (v - w^*, u^*_x) + (u^*, (v - w^*)_x) \\ &= \frac{1}{2}(c^{-1}(v - w^*), v - w^*) \geq 0\,. \end{aligned}$$

On the other hand, for any u such that $u(0) = 0$, we have

$$L(u, w^*) = \frac{1}{2}(c^{-1}w^*, w^*) + (u, w^*_x + f) = \frac{1}{2}(c^{-1}w^*, w^*)\,,$$

so we know that $L(u, w^*)$ is constant with respect to u. This proves both w^* and u^* are the desired solutions to the optimization problems in (1.36).

Next we show that if w^* and u^* are functions such that $w^*(1) = 0$, $u(0) = 0$, and are the minimizer and the maximizer of the optimization problems in (1.36), then they are also the solutions to (1.37).

To do so, we define (for simplicity we drop the index $*$ in u^* and w^*)

$$F(t) = L(u, w + tv) \quad \text{for any } v\ .$$

As w is the minimizer of $L(u, w)$, we know

$$F(t) = L(u, w + tv) \geq L(u, w) = F(0) \quad \forall\, t \in \mathbb{R}^1,$$

so $t = 0$ is a minimizer for $F(t)$, thus

$$F'(0) = 0\ .$$

Now by definition,

$$\begin{aligned}
F(t) - F(0) =& L(u, w + tv) - L(u, w)\\
=& \frac{1}{2}\int_0^1 \frac{1}{c}(w + tv)^2 dx + \int_0^1 u\big(\frac{d(w + tv)}{dx} + f\big)dx\\
&- \Big\{\frac{1}{2}\int_0^1 \frac{1}{c}w^2 dx + \int_0^1 u\big(\frac{dw}{dx} + f\big)\Big\}dx\ ,
\end{aligned}$$

or

$$F(t) - F(0) = t\int_0^1 \big(\frac{1}{c(x)}wv + u\frac{dv}{dx}\big)dx + \frac{1}{2}t^2\int_0^1 \frac{1}{c(x)}v^2 dx,$$

therefore

$$0 = F'(0) = \int_0^1 \Big(\frac{1}{c(x)}wv + u\frac{dv}{dx}\Big)dx\ .$$

Integration by parts gives

$$\int_0^1 \Big(\frac{1}{c(x)}w - \frac{du}{dx}\Big)v dx + uv(x)\Big|_{x=0}^{x=1} = 0 \quad \forall v$$

which implies

$$\frac{1}{c(x)}w = \frac{du}{dx} \quad \text{or} \quad w = c(x)\frac{du}{dx}\ . \tag{1.38}$$

Thus we get back the original physical law $w = c(x)\frac{du}{dx}$.

On the other hand, we can find the maximizer of u, that gives the condition:

$$-\frac{dw}{dx} = f(x)\,. \tag{1.39}$$

This proves the desired results.

From the equations (1.38)-(1.39, we see that u satisfies

$$-\frac{d}{dx}\big(c(x)\frac{du}{dx}\big) = f(x)\,, \quad 0 < x < 1.$$

This means that the Lagrangian multiplier u is actually the displacement function.