Lecture 8: Recap:

Definition: (Discrete Fourier Transform) Given $f_0, f_1, \dots, f_{n-1} \in \mathbb{C}$, then the discrete Fourier Transform (DFT) is defined as:

$$\vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \in \mathbb{C}^n \quad \text{where} \quad c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j \, e^{-i\left(\frac{2jk\pi}{n}\right)} \quad \text{for } k = 0, 1, 2, \dots, n-1$$

The inverse discrete Fourier Transform recovers the original signal:

$$f_j = \sum_{k=0}^{n-1} c_k \, e^{i\left(\frac{2jk\pi}{n}\right)} \quad \text{for } j = 0, 1, 2, \dots, n-1$$

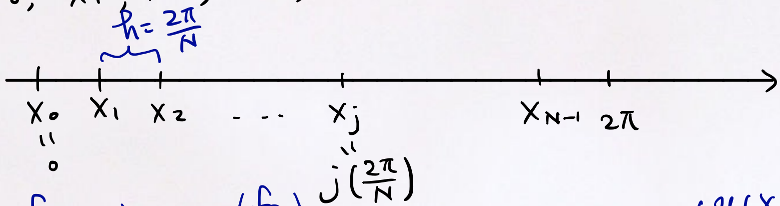
Recall:

Consider: $\dfrac{d^2u}{dx^2} = f$ for $x \in [0, 2\pi]$ with periodic boundary condition.

$u(0) = u(2\pi)$

Suppose f is measured only at N discrete points:

$x_0, x_1, x_2, \ldots, x_{N-1}$



(unknown)

Let $\vec{f} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} \in \mathbb{R}^N$ and $\vec{u} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_{N-1}) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} \in \mathbb{R}^N$

Then: $$\begin{pmatrix} u''(x_0) \\ u''(x_1) \\ \vdots \\ u''(x_{N-1}) \end{pmatrix} \approx \tilde{D}\vec{u} \quad \text{where} \quad \tilde{D} = \frac{1}{h^2}\underbrace{\begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & 0 & \\ & 1 & \ddots & \ddots & \\ & 0 & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{pmatrix}}_{M_{N\times N}(\mathbb{R})}$$

(using the fact $u_0 = u_N$, $u_{-1} = u_{N-1}$)

Conclusion:

$$\boxed{\frac{d^2u}{dx^2} = f \quad \text{can be discretized as} \quad \boxed{\tilde{D}\vec{u} = \vec{f}} \quad \text{(Linear System)}}$$

Last time:

$\overrightarrow{e^{ikx}}$ is an eigenvector of $\tilde{D}$ for $k = 0, 1, 2, \ldots, N-1$

Claim: $\left\{ \overrightarrow{e^{ikx}} \right\}_{k=0}^{N-1}$ is a basis of $\mathbb{C}^N$ (consisting of eigenvectors)

Pf:

$$\begin{pmatrix} \vert & \vert & & \vert \\ \overrightarrow{e^{i0x}} & \overrightarrow{e^{i(1)x}} & \cdots & \overrightarrow{e^{i(N-1)x}} \\ \vert & \vert & & \vert \end{pmatrix} = \underset{\sim}{A_w} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{N-1} \\ \vdots & & & \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{pmatrix} ; \quad \omega = e^{i\frac{2\pi}{N}}$$

Claim: $\operatorname{Rank}(\tilde{D}) = N-1$ and null space of D is:

$$N(\tilde{D}) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

Pf: D^2 has $N-1$ distinct non-zero eigenvalues.

$\therefore \operatorname{Rank}(\tilde{D}) = N-1$.

$N(\tilde{D})$ = eigenspace associated to zero eige value $= \lambda_0$.

The only eigenvector of $\tilde{D}$ with eigenvalue $= 0$ is: $\overrightarrow{e^{i0x}} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

$\therefore N(\tilde{D}) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\}$.

Claim: If $\vec{u}_1$ and $\vec{u}_2$ are both solutions of $\tilde{D}\vec{u} = \vec{f}$, then:

$$\vec{u}_1 = \vec{u}_2 + c\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} \text{ for some constant } c.$$

Proof: $\tilde{D}\vec{u}_1 = \vec{f}$; $\tilde{D}\vec{u}_2 = \vec{f}$ $\quad \therefore\ D(\vec{u}_1 - \vec{u}_2) = 0$

$$\therefore\ \vec{u}_1 - \vec{u}_2 \in N(\tilde{D}) = \text{span}\left\{\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\right\}.$$

Thus, $\vec{u}_1 = \vec{u}_2 + c\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$ for some c.

Numerical Spectral method

Since $\{\overrightarrow{e^{ikx}}\}_{k=0}^{N-1}$ is a basis. We can write:

$$\vec{u} = \sum_{k=0}^{N-1} \hat{u}_k \overrightarrow{e^{ikx}} \quad \text{and} \quad \vec{f} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

$\mathbb{C}$

In other words, for each j, $f_j = f(x_j) = \sum_{k=0}^{N-1} \hat{f}_k \left(\overrightarrow{e^{ikx}}\right)_j$

$$= \sum_{k=0}^{N-1} \hat{f}_k e^{ikx_j}$$

(DFT!)

$\therefore$ $\hat{f}_k$ can be determined by DFT.

To solve $\frac{d^2u}{dx^2} = f$, we approximate it by

$$\tilde{D}\vec{u} = \vec{f}.$$

Now, $\tilde{D}\vec{u} = \vec{f}$ becomes:

$$\tilde{D}\left(\sum_{k=0}^{N-1} \hat{u}_k \overrightarrow{e^{ikx}}\right) = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

$$\Longleftrightarrow \sum_{k=0}^{N-1} \hat{u}_k \underbrace{\tilde{D}\overrightarrow{e^{ikx}}}_{=\,(-\lambda k^2)\overrightarrow{e^{ikx}}} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

$$\Longleftrightarrow \sum_{k=0}^{N-1} \hat{u}_k (-\lambda k^2)\overrightarrow{e^{ikx}} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

Comparing coefficients, we get

$$\underbrace{-\lambda k^2}_{\text{known}} \underbrace{\hat{u}_k}_{\text{unknown}} = \underbrace{\hat{f}_k}_{\text{known}} \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

(algebraic equation)