Lecture 6:

Recall:

Definition: For a given smooth $f(x)$ defined on $(-\infty, \infty)$, the Fourier transform of f is a function $\hat{f}$ depending on the frequency k:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \quad , \quad -\infty < k < \infty$$

The inverse Fourier Transform of $\hat{f}(k)$ recovers the original function $f(x)$:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk \quad , \quad -\infty < x < \infty$$

Recall:

Important properties of Fourier Transform:

① $\widehat{\alpha f}(k) = \alpha \hat{f}(k)$ for all $\alpha \in \mathbb{R}$

② $\widehat{f+g}(k) = \hat{f}(k) + \hat{g}(k)$

③ $\widehat{\frac{df}{dx}}(k) = (ik)\hat{f}(k)$

Example: Define delta function $\delta(x)$ as a "function" such that:

$$\int_{-\infty}^{\infty} g(x)\,\delta(x)\,dx = g(0).$$

$\delta(x)$ is like a "spike", that's non-zero ONLY at $x=0$ and $\int \delta(x)\,dx = 1$!!

Then, the Fourier Transform of $\delta(x)$ is:

$$\hat{f}(k) = \int_{-\infty}^{\infty} \delta(x)\,e^{-ikx}\,dx = e^{-ik(0)} = 1 \quad \text{for all } k.$$

Example: Let $f(x) = 1$.

Then:
$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx}\,dx = \int_{0}^{\infty} e^{-ikx}\,dx + \int_{-\infty}^{0} e^{-ikx}\,dx$$

$$\approx \lim_{a\to 0^+} \left\{ \int_0^{\infty} e^{-ax} e^{-ikx}\,dx + \int_{-\infty}^{0} e^{ax} e^{-ikx}\,dx \right\} \quad (a>0)$$

$$= \lim_{a\to 0^+} \left\{ \frac{1}{a+ik} + \frac{1}{a-ik} \right\} = \begin{cases} 0 & k \neq 0 \\ ?? & k = 0 \end{cases}.$$

What's $\hat{f}(0)$?? Note that $\hat{f}(k)$ looks like a "spike" (or delta function). Let $\hat{f}(k) = \alpha\,\delta(k)$. Then, by the inverse Fourier Transform, we have:

$$1 = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{\alpha\,\delta(k)}{\underset{\shortparallel}{\hat{f}(k)}}\, e^{ikx}\,dk = \frac{\alpha}{2\pi}. \qquad \therefore\ \alpha = 2\pi.$$

$$\therefore\quad \hat{f}(k) = 2\pi\,\delta(k).$$

One more useful property of Fourier Transform

Let $u(x) = \int_{-\infty}^{\infty} G(x-y)\, h(y)\, dy$. We often write it as:

$$u(x) = (G * h)(x) \quad \text{(Convolution of } G \text{ and } h \text{)}$$

Then: $\hat{u}(k) = \hat{G}(k)\, \hat{h}(k)$.

Proof:

$$\hat{u}(k) = \int_{-\infty}^{\infty} u(x)\, e^{-ikx}\, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x-y)\, h(y)\, e^{-ikx}\, dy\, dx$$

$$= \int_{-\infty}^{\infty} h(y) \left(\int_{-\infty}^{\infty} G(x-y)\, e^{-ikx}\, dx \right) dy$$

$$= \int_{-\infty}^{\infty} h(y)\, e^{-iky}\, dy \int_{-\infty}^{\infty} G(x')\, e^{-ikx'}\, dx' \quad (\text{let } x-y=x')$$

$$= \hat{G}(k)\ \hat{h}(k)$$

Example: Consider: $-\dfrac{d^2u}{dx^2} + a^2 u = h(x)$, $-\infty < x < \infty$, $a > 0$

Apply Fourier transform on both sides:

$$-(ik)^2\hat{u}(k) + a^2\hat{u}(k) = \hat{h}(k) \quad \text{(algebraic eqt)}$$

$$\therefore (a^2 + k^2)\,\hat{u}(k) = \hat{h}(k) \Rightarrow \hat{u}(k) = \frac{\hat{h}(k)}{(a^2+k^2)}$$

Consider $G(x)$ where $\hat{G}(k) = \dfrac{1}{a^2+k^2}$. Then:

$$\hat{u}(k) = \hat{h}(k)\,\hat{G}(k) \Rightarrow u(x) = G * h(x)$$

$$= \int_{-\infty}^{\infty} G(x-y)\,h(y)\,dy$$

Now, consider $f(x) = e^{-a|x|}$ Then:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx = \int_0^{\infty} e^{-(a+ik)x}\,dx + \int_{-\infty}^{0} e^{(a-ik)x}\,dx$$

$$= \frac{1}{a+ik} + \frac{1}{a-ik} = \frac{2a}{a^2+k^2}$$

$\therefore \widehat{\frac{1}{2a} f}(k) = \frac{1}{a^2 + k^2}$ $\therefore G(x) = \frac{1}{2a} e^{-a|x|}$.

All together, $u(x) = G * h(x) = \frac{1}{2a} \int_{-\infty}^{\infty} e^{-a|x-y|} h(y)\, dy$

<u>Remark:</u> $\widehat{G * h}(k) = \hat{G}(k)\, \hat{h}(k)$ and $\widehat{\frac{du}{dx}}(k) = ik\, \hat{u}(k)$

are two most important properties for solving differential eqt using Fourier Transform!!

Question: Extension to discrete case (Computational Math.)

Answer: Discrete Fourier Transform

Goal: ① Define discrete Fourier Transform (DFT)

② Use DFT to solve discretized differential eqt.

Definition: (Discrete Fourier Transform) Given $f_0, f_1, \ldots, f_{n-1} \in \mathbb{C}$, then the discrete Fourier Transform (DFT) is defined as:

$$\vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \in \mathbb{C}^n \quad \text{where} \quad c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j \, e^{-i\left(\frac{2jk\pi}{n}\right)}$$

for $k = 0, 1, 2, \ldots, n-1$

The inverse discrete Fourier Transform recovers the original signal:

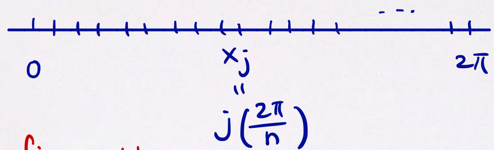
$$f_j = \sum_{k=0}^{n-1} c_k \, e^{i\left(\frac{2jk\pi}{n}\right)} \quad \text{for } j = 0, 1, 2, \ldots, n-1$$

Motivation of the definition:

Let $f(x)$ be defined on $[0, 2\pi]$. Approximate $f(x)$ by:

$$F_n(x) = \sum_{k=0}^{n-1} c_k e^{ikx}, \quad x \in [0, 2\pi] \quad \text{such that:}$$

$$F_n(x_j) = f(x_j) := f_j, \quad x_j = j\left(\frac{2\pi}{n}\right)$$



Then: $\vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}$ given by DFT satisfies the above.

<u>Remark:</u> DFT approximates $f(x)$ as a linear combination of basis functions $\{e^{ikx}\}_{k=0}^{n-1}$.