Lecture 3:

Another useful technique: Separation of variables

Consider a heat equation (on a unit circle):

$$u_t = u_{xx}, \quad x \in [0, 2\pi], \quad t \geq 0$$

Subject to: $\begin{cases} u(0,t) = u(2\pi, t) & \text{(periodic condition)} \\ u(x,0) = \sin x & \text{(initial condition)} \end{cases}$

<u>Strategy:</u> Let $u(x,t) = X(x)\,T(t)$.

$$u_t = u_{xx} \Rightarrow X(x)\,T'(t) = X''(x)\,T(t)$$

$$\therefore \frac{X''}{X} = \frac{T'}{T} = \lambda \leftarrow \text{some constant}$$

In particular, $T' = \lambda T$ for some constant λ.

$$\Rightarrow \frac{d}{dt}(\ln T) = \lambda \Rightarrow \ln T = \lambda t + C_0 \Rightarrow \boxed{T = Ce^{\lambda t}}$$

for some constant C and λ.

For X, since $u(x,0) = \sin x$. We may guess $X(x) = \sin x$

Note that $X'' = (-1)\sin x = (-1)X$. $\therefore \lambda = -1$.

Hence a possible solution is of the form:

$$u(x,t) = Ce^{-t}\sin x$$

$$\because u(x,0) = \sin x = C\sin x \Rightarrow C = 1.$$

$\therefore u(x,t) = e^{-t}\sin x$ is a solution.
(multi-variable)

Remark: Separate $u(x,t) = X(x)\,T(t)$ $\rightsquigarrow$ PDE converted to 2 ODEs (single variable)

single variable function

Spectral method

We'll discuss : (1) Analytic (Fourier) Spectral method

(2) Numerical Spectral method

Main idea: Consider : $Lu(x) = g(x)$ such that

u and g are periodic functions $\left(\text{i.e. } \begin{array}{l} u(x+2\pi) = u(x) \\ g(x+2\pi) = g(x) \end{array} \right)$

where L is a linear differential operator $\left(\text{e.g. } L = \dfrac{d^2}{dx^2} ;\quad L = \dfrac{d^2}{dx^2} + \dfrac{d}{dx} \text{ etc} \right)$

$\left(\text{e.g. if } L = \dfrac{d^2}{dx^2} + \dfrac{d}{dx}, \text{ then } Lu(x) = \dfrac{d^2u}{dx^2}(x) + \dfrac{du}{dx}(x) \right)$

L is linear means : $L\big(u(x) + a\,v(x)\big) = Lu(x) + a\,Lv(x)$, $a \in \mathbb{R}$

Assume that $\{\phi_n(x)\}_{n=1}^{\infty}$ are functions such that:

(1) $\phi_n(x)$ is periodic;

(2) $L\phi_n(x)$ is a linear combination $\{\phi_n(x)\}_{n=1}^{\infty}$

Assume: $u(x) \approx \sum_{k=0}^{N} a_k \phi_k(x)$ and $g(x) \approx \sum_{k=0}^{N} b_k \phi_k(x)$

(Note: in solving the differential equation, a_k's are unknown, b_k's are known)

Then: $\phi_n(x)$ is called the basis functions for the differential equation $Lu(x) = g(x)$.

For the ease of explanation, suppose $L\phi_n(x) = \lambda_n \phi_n(x)$.

($\phi_n(x)$ is an eigenfunction of L)

Goal: Find a_k's solving $Lu(x) = g(x)$.

Then: $Lu(x) = g(x)$ implies: $L\left(\sum_{k=0}^{N} a_k \phi_k(x)\right) = \sum_{k=0}^{N} b_k \phi_k(x)$

$$\Rightarrow \quad \sum_{k=0}^{N} a_k L\phi_k(x) = \sum_{k=0}^{N} b_k \phi_k(x)$$

$$\Rightarrow \quad \sum_{k=0}^{N} a_k \lambda_k \phi_k(x) = \sum_{k=0}^{N} b_k \phi_k(x).$$

Comparing coefficients: $a_k \lambda_k = b_k$ (algebraic equation)

$$\therefore \quad a_k = \frac{b_k}{\lambda_k}$$

Thus, the solution is: $u(x) = \sum_{k=0}^{N} \left(\frac{b_k}{\lambda_k}\right) \phi_k(x)$.

Remark: 1. By writing $u(x)$ as linear combination of basis functions (eigenfunctions), complicated differential equation can be converted to differential equation.

2. Spectral method is related to eigenvalues and eigenfunctions of some differential operators (e.g. $\sin x$ is eigenfunction of $\frac{d^2}{dx^2}$)

It's called Spectral decomposition of differential operator.

Example: Consider $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin x + 2\cos x$ where $y(0) = y(2\pi)$ (periodic). Find a possible solution of the 2nd order ODE.

Note that: $L = \frac{d^2}{dx^2} + \frac{d}{dx}$ in our case.

Solution: We first need to construct $\{\phi_n(x)\}_{n=1}^{\infty}$.

Note that $\phi_n(x) = \sin nx$ or $\cos nx$ are all possible choices.

$\therefore$ take $\{\phi_n(x)\}_{n=1}^{\infty} = \{\sin nx, \cos nx\}_{n=0}^{\infty}$

e.g. $L(\sin nx) = -n^2 \sin nx + n\cos nx$ (linear combination of $\{\phi_n(x)\}_{n=1}^{\infty}$)

Let $y(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$

Then: $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin x + 2\cos x$ implies:

$$\sum_{n=1}^{N}\left(-a_n n^2 \cos nx - b_n n^2 \sin nx\right) + \sum_{n=1}^{N}\left(-n a_n \sin nx + n b_n \cos nx\right) = \sin x + 2\cos x$$

$$\Rightarrow \sum_{n=1}^{N}\left[(n b_n - n^2 a_n)\cos nx - (n a_n + n^2 b_n)\sin nx\right] = \sin x + 2\cos x$$

Comparing coefficient: $b_1 - a_1 = 2$, $a_1 + b_1 = -1$ $\Rightarrow$ $b_1 = 1/2$, $a_1 = -3/2$ (Algebraic eqt)

$a_k = b_k = 0$ otherwise

$\therefore$ A possible solution is: $y(x) = -\frac{3}{2}\cos x + \frac{1}{2}\sin x$

Example: Consider: $u_t = u_{xx}$, $x \in [0, 2\pi]$ such that

$$u(0,t) = u(2\pi, t) \text{ (periodic)}$$
$$u(x,0) = f(x) \text{ (initial condition)}$$

Solution: Let $u(x,t) = X(x)\,T(t)$

Consider $L = \frac{\partial^2}{\partial x^2}$. Construct $\{\phi_n(x)\}_{n=1}^{\infty} = \{\cos nx, \sin nx, \cancel{e^{nx}}\}_{n=0}^{\infty}$

But: $u(0,t) = u(2\pi,t) \Rightarrow X(0)\,T(t) = X(2\pi)\,T(t)$

$\Rightarrow X(0) = X(2\pi)$

X must be periodic. $\therefore$ e^{kx} CANNOT be the choice!!

Let $u(x,t) = \sum_{n=1}^{N} a_n(t) \cos nx + b_n(t) \sin nx$

$$u_t = u_{xx} \Rightarrow \sum_{n=1}^{N} a_n'(t) \cos nx + b_n'(t) \sin nx = \sum_{n=1}^{N} (-n^2)\,a_n(t) \cos nx + (-n^2)\,b_n(t) \sin nx$$

Comparing coefficients: $a_n'(t) = -n^2 a_n(t)$ and $b_n'(t) = -n^2 b_n(t)$.

Solving $a_n'(t) = -n^2 a_n(t) \Rightarrow a_n(t) = a_n e^{-n^2 t}$ $(a_n \in \mathbb{R})$

Similarly, $b_n(t) = b_n e^{-n^2 t}$ $(b_n \in \mathbb{R})$

$$\therefore \quad u(x,t) = \sum_{n=1}^{N} a_n e^{-n^2 t} \sin nx + \sum_{n=1}^{N} b_n e^{-n^2 t} \cos nx$$

How to determine a_k and b_k ?? Initial condition: $u(x,0) = f(x)$.

Suppose $f(x) = \sum_{k=0}^{\infty} c_k \cos kx + d_k \sin kx$.

Then: $u(x,0) = f(x)$ implies:

$$\sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx = \sum_{k=0}^{\infty} c_k \cos kx + d_k \sin kx$$

Comparing coefficients:

$$a_k = c_k$$
$$b_k = d_k$$

(Algebraic eqt).

Question: Given $f(x)$, how to find a_k and b_k such that $f(x) = \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx$?

(Fourier analysis problem)

Note that:

$$\int_0^{2\pi} \cos kx \cos mx \, dx = \begin{cases} 2\pi & \text{if } k = m = 0 \\ \pi & \text{if } k = m \neq 0 \\ 0 & \text{if } k \neq m \end{cases}$$

e.g. $\int_0^{2\pi} \cos kx \cos kx \, dx = \int_0^{2\pi} \frac{1 + \cos(2kx)}{2} dx = \pi$

Also,

$$\int_0^{2\pi} \sin kx \sin mx \, dx = \begin{cases} 2\pi & \text{if } k = m = 0 \\ \pi & \text{if } k = m \neq 0 \\ 0 & \text{if } k \neq m \end{cases}$$

$$\int_0^{2\pi} \sin kx \cos mx \, dx = 0 .$$

If $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx.$

For $m > 0$, $\int_0^{2\pi} f(x) \cos mx \, dx = \pi a_m$

$\therefore \quad a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx \, dx$

$\int_0^{2\pi} f(x) \sin mx \, dx = \pi b_m$

$\therefore \quad b_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx \, dx.$

Also, $\int_0^{2\pi} f(x) \, dx = a_0 (2\pi) \Rightarrow a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx.$

$\therefore$ All a_k, b_k can be computed!!

Example: Consider $u_t - \alpha u_{xx} = h(x,t)$ such that:

$u(0,t) = u(2\pi, t) = 0$ and $u(x,0) = f(x)$.

Assuming that $h(x,t) = \sum_{k=1}^{\infty} k^2 t \sin kx$ and $f(x) = \sum_{k=1}^{\infty} k \sin kx$

Solution: This time, we consider $L = \frac{d^2}{dx^2}$.

Then: we choose $\{\phi_n(x)\}_{n=1}^{\infty} = \{\cos nx, \sin nx\}_{n=1}^{\infty}$.

We assume:

$$u(x,t) = \sum_{n=1}^{\infty} \cancel{a_n(t)\cos nx} + b_n(t)\sin nx$$

Note that $u(0,t) = u(2\pi,t) = 0$, we can remove terms with $\cos nx$

$\therefore$ we assume $u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx$.

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = h(x,t)$$

$$\Rightarrow \sum_{k=1}^{\infty} b_k'(t) \sin kx + \alpha b_k(t) k^2 \sin kx = \sum_{k=1}^{\infty} k^2 t \sin kx$$

Comparing coefficients :

$$b_k'(t) + \alpha k^2 b_k(t) = k^2 t$$

Also, $u(x,0) = \sum_{k=1}^{\infty} b_k(0) \sin kx = f(x) = \sum_{k=1}^{\infty} k \sin kx$

Comparing coefficients :

$$b_k(0) = k.$$

$\therefore$ we have:

$$\begin{cases} b_k'(t) + \alpha k^2 b_k(t) = k^2 t \\ b_k(0) = k \end{cases}$$

Can be solved using integrating factor technique:

Let $M(t) = e^{\int \alpha k^2 dt}$. Multiply both sides by $M(t)$ etc. . —

Answer: $$b_k(t) = e^{-\alpha k^2 t}\left(k + \int_0^t k^2 s\, e^{\alpha k^2 s}\, ds \right)$$

(exercise!)