Lecture 15:

Splitting choice 3: Successive overrelaxation method (SOR)

Consider the iterative scheme:

(Suppose $A = L + D + U$)

$$L\vec{x}^{k+1} + D\vec{y}^{k+1} + U\vec{x}^{k} = \vec{b} \quad \text{———} \quad (*)$$

$$\vec{x}^{k+1} = \vec{x}^{k} + \omega(\vec{y}^{k+1} - \vec{x}^{k}) \quad \text{———} \quad (**)$$

$$\Longleftrightarrow \quad \vec{y}^{k+1} = \frac{1}{\omega}(\vec{x}^{k+1} + (\omega - 1)\vec{x}^{k})$$

Putting (**) into (*):

$$\left(L + \frac{1}{\omega}D\right)\vec{x}^{k+1} + \frac{1}{\omega}(\omega U + (\omega - 1)D)\vec{x}^{k} = \vec{b}$$

or $$\overbrace{\left(L + \frac{1}{\omega}D\right)}^{N}\vec{x}^{k+1} = \overbrace{\left(\frac{1}{\omega}D - (D+U)\right)}^{P}\vec{x}^{k} + \vec{b} \quad (\text{SOR})$$

Remark: • SOR is equivalent to:

$$A = N - P = \underbrace{\left(L + \tfrac{1}{\omega}D\right)}_{N} - \underbrace{\left(\tfrac{1}{\omega}D - (D+U)\right)}_{P}$$

Or equivalently, $A = (a_{ij})$

$$\begin{cases} a_{11} y_1^{k+1} + a_{12} x_2^k + \dots + a_{1n} x_n^k = b_1 & \text{for } x_1^{k+1} = x_1^k + \omega(y_1^{k+1} - x_1^k) \\ a_{21} x_1^{k+1} + a_{22} y_2^{k+1} + a_{23} x_3^k + \dots + a_{2n} x_n^k = b_2 & \text{for } x_2^{k+1} = x_2^k + \omega(y_2^{k+1} - x_2^k) \\ \vdots & \\ a_{n1} x_1^{k+1} + a_{n2} x_2^{k+1} + \dots + a_{nn} y_n^{k+1} = b_n & \text{for } x_n^{k+1} = x_n^k + \omega(y_n^{k+1} - x_n^k) \end{cases}$$

• SOR = Gauss-Seidel if $\omega = 1$.

Another condition for the convergence (useful for analyzing SOR)

Let $A = N - P$ (N is invertible)

Iterative scheme : $N\vec{x}^{k+1} = P\vec{x}^{k} + \vec{b}$

Theorem: (Household - John) Suppose A and $(N^* + N - A)$ are self-adjoint positive-definite matrices, then the iterative scheme converges.

<u>Proof:</u> Consider $M = N^{-1}P = N^{-1}(N-A) = I - N^{-1}A$.

Suffice to show that all eigenvalues of M satisfy $|\lambda| < 1$ (λ can be complex). Let λ be an eigenvalue associated to the eigenvector $\vec{x}$. Then:

$$M\vec{x} = \lambda\vec{x} \Rightarrow (I - N^{-1}A)\vec{x} = \lambda\vec{x} \Rightarrow (N-A)\vec{x} = \lambda N\vec{x}$$

$$\Rightarrow (1-\lambda)N\vec{x} = A\vec{x}$$

Note that $\lambda \neq 1$. Otherwise 0 is an eigenvalue of A. Contradiction to the fact that A is positive definite.

Multiply $\vec{x}^*$ on both sides:

$$(1-\lambda)\vec{x}^* N\vec{x} = \vec{x}^* A\vec{x} \Rightarrow \vec{x}^* N\vec{x} = \frac{1}{(1-\lambda)}\vec{x}^* A\vec{x} \quad \text{— (1)}$$

Take conjugate transpose on both sides:

$$(1-\bar{\lambda})\,\vec{x}^* N^* \vec{x} = \vec{x}^* A^* \vec{x} = \vec{x}^* A \vec{x}$$

$$\Rightarrow \quad \vec{x}^* N^* \vec{x} = \frac{1}{(1-\bar{\lambda})}\,\vec{x}^* A \vec{x} \quad \text{— (2)}$$

(1) + (2) − $\vec{x}^* A \vec{x}$ on both sides:

$$\vec{x}^*(N + N^* - A)\,\vec{x} = \left(\frac{1}{1-\lambda} + \frac{1}{1-\bar{\lambda}} - 1\right)\vec{x}^* A \vec{x}$$

$$= \frac{1-|\lambda|^2}{|1-\lambda|^2}\,\vec{x}^* A \vec{x}$$

By assumption, A and $N+N^*-A$ are both positive definite.

We have: $\vec{x}^* A \vec{x} > 0$ and $\vec{x}^*(N+N^*-A)\,\vec{x} > 0$

Hence, $1-|\lambda|^2 > 0$ and $|\lambda| < 1$.

$\therefore$ $\rho(M) < 1$ and the iterative scheme converges.

Condition for convergence

Theorem: The neccessary condition (not sufficient) for SOR to converge is $0 < \omega < 2$.

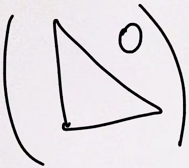
Proof: Consider: $\det(N^{-1}P)$ = product of eigenvalues of $N^{-1}P$

$$= \det\left(\left(L + \frac{1}{\omega}D\right)^{-1}\left(\frac{1}{\omega}D - (D+U)\right)\right)$$

$$= \det\left(\left(\frac{1}{\omega}D\right)^{-1}\right)\det\left(\frac{1}{\omega}D - D\right)$$

$$= \det\left(\cancel{\omega}\,\cancel{D^{-1}}\right)\det\left(\frac{1}{\cancel{\omega}}(1-\omega)\cancel{D}\right)$$

$$= \det\left((1-\omega)I\right) = (1-\omega)^n$$



$$\therefore\ \left|\det(N^{-1}P)\right| = \prod_{i=1}^{n} |\lambda_i| = |1-\omega|^n$$

λ_i = eigenvalue of $N^{-1}P$

$$|1-\omega|^n = \prod_{i=1}^{n}|\lambda_i| \le \left(\max_i |\lambda_i|\right)^n = \rho(N^{-1}P)^n$$

$$\rho(N^{-1}P) \ge |1-\omega|$$

Now, SOR converges iff $\rho(N^{-1}P) < 1$

$$\therefore \ |1-\omega| \le \rho(N^{-1}P) < 1 \implies 0 < \omega < 2.$$

<u>Remark:</u> In general, the convergence and convergence rate depends on:

$$\rho(N^{-1}P) < 1$$

Example: Go back: $A\vec{x} = \begin{pmatrix} 10 & 1 \\ 1 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 21 \end{pmatrix} := \vec{b}$.

Recall: $\underset{N_J^{-1}P_J}{\overset{\parallel}{\rho(M_J)}} = \frac{1}{10}$; $\underset{N_{GS}^{-1}P_{GS}}{\overset{\parallel}{\rho(M_{G-S})}} = \frac{1}{100}$

For SOR,
$$\begin{aligned} M_{SOR} &= \left(L + \frac{1}{\omega}D\right)^{-1}\left(\frac{1}{\omega}D - (D+U)\right) \\ &= \left(\frac{1}{\omega}(D+\omega L)\right)^{-1}\left(\frac{1}{\omega}(D - \omega(D+U)\right) \\ &= \cancel{\omega}\,(D+\omega L)^{-1}\,\cancel{\left(\frac{1}{\omega}\right)}\,((1-\omega)D - \omega U) \\ &= \end{aligned}$$

$$\therefore \quad M_{SOR} = \begin{pmatrix} 1-\omega & -\omega/10 \\ \frac{-\omega(1-\omega)}{10} & \frac{\omega^2}{100} + (1-\omega) \end{pmatrix}$$

$\therefore$ Characteristic polynomial of M_{SOR}:

$$\left[(1-\omega)-\lambda\right]\left[\frac{\omega^2}{100}+1-\omega-\lambda\right]-\frac{\omega^2(1-\omega)}{100}=0 \quad (\ast\ast)$$

Solving it:

$$\lambda=(1-\omega)+\frac{\omega^2}{200}\pm\frac{\omega}{20}\sqrt{4(1-\omega)+\frac{\omega^2}{100}}$$

<u>Remark:</u> • Adjusting ω gives different eigenvalues and different $\rho(M_{SOR})$

• Put $\omega=1$ (G-S), $\lambda=0$, $\lambda=\frac{1}{100}$

• Choose ω such that $(\ast\ast)$ has a repeated root. That is,

$$4(1-\omega)+\frac{\omega^2}{100}=0 \Rightarrow \omega=1.002512579$$

Then: $\lambda=(1-\omega)+\frac{\omega^2}{200}=(1-\omega)-2(1-\omega)=\omega-1=0.002512579$

$\rho(M_J)=0.1$, $\rho(M_{G\text{-}S})=0.01$, $\rho(M_{SOR})=0.002512579$

Remark: For some special matrix, optimal ω can be explicitly found.

Theorem: If A is strictly diagonally dominant (SDD), then SOR converges if $0 < \omega \leq 1$.

Proof: We need to show $\rho(M_{SOR}) < 1$ if $0 < \omega \leq 1$.

We'll show it by contradiction.

Suppose $\exists$ eigenvalue λ such that $|\lambda| \geq 1$. Then:

$$\det(\lambda I - M_{SOR}) = 0$$

$$\therefore \det\left(\lambda I - (D + \omega L)^{-1}\left((1-\omega)D - \omega U\right)\right) = 0$$

$$\Rightarrow \det\Big(\underset{\underset{0}{\neq}}{\lambda}\,(D+\omega L)^{-1}\underbrace{\left((D+\omega L) - \frac{1}{\lambda}\left((1-\omega)D - \omega U\right)\right)}_{C}\Big) = 0$$

$$\Rightarrow \det(C) = 0 \quad (\because \lambda \neq 0,\ (D + \omega L) \text{ is invertible})$$

Note: $\omega\left(1-\frac{1}{|\lambda|}\right) \le \left(1-\frac{1}{|\lambda|}\right) \Rightarrow \left(1-\frac{1}{|\lambda|}(1-\omega)\right) \ge \omega$

Now,

$$|C_{ii}| = \left|1-\frac{1}{\lambda}(1-\omega)\right| |a_{ii}| \ge \left[1-\frac{1}{|\lambda|}(1-\omega)\right]|a_{ii}|$$

$$\ge \omega|a_{ii}| > \omega\sum_{\substack{j=1\\ j\ne i}}^{n}|a_{ij}| \ge \underbrace{\omega\sum_{j=1}^{i-1}|a_{ij}| + \frac{\omega}{|\lambda|}\sum_{j=i+1}^{n}|a_{ij}|}_{=\sum_{\substack{j=1\\ j\ne i}}^{n}|C_{ij}|}$$

$$|C_{ii}| > \sum_{\substack{j=1\\ j\ne i}}^{n}|C_{ij}| \quad \text{for all } i$$

$\therefore$ C is SDD $\Rightarrow$ C is non-singular!!

$\Rightarrow \det(C) \neq 0$

Contradiction.

$$\boxed{\underbrace{\left((D+\omega L) - \frac{1}{\lambda}\left((1-\omega)D - \omega U\right)\right)}_{=\,C}}$$

Optimal parameter ω_{opt} for SOR method

Definition: Consider the system $A\vec{x} = \vec{b}$. Let $A = \overset{\text{lower}}{L} + \overset{\text{diagonal}}{D} + \overset{\text{upper}}{U}$.

If the eigenvalues of : $\alpha D^{-1} L + \frac{1}{\alpha} D^{-1} U$ $(\alpha \neq 0)$ are independent of α. Then, the matrix A is said to be consistently ordered.

Example of consistently ordered matrices

1. Tridiagonal matrix :

$$\begin{pmatrix} \lambda_1 & * & & 0 \\ * & \lambda_2 & * & \\ & \ddots & \ddots & * \\ 0 & & * & \lambda_n \end{pmatrix}$$

2. Block tridiagonal matrix :

$$\begin{pmatrix} \boxed{D_1} & T_{12} & & 0 \\ T_{21} & \boxed{D_2} & T_{23} & \\ & \ddots & \ddots & \ddots \\ 0 & & & \boxed{D_p} \end{pmatrix}$$

where D_i = diagonal matrix

Theorem: [D. Young] Assume:

1. $0 < \omega < 2$
2. $M_J = N_J^{-1} P_J$ has only real eigenvalues
3. $\beta \overset{def}{=} \rho(M_J) < 1$
4. A is consistently ordered.

Then: $\rho(M_{SOR}) < 1$

Also, the optimal parameter ω_{opt} for fastest convergence

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \beta^2}} \quad \text{and}$$

$$\rho(M_{SOR,\omega_{opt}}) = \omega_{opt} - 1$$

Example: $A\vec{x} = \begin{pmatrix} 10 & 1 \\ 1 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 21 \end{pmatrix}$

Then: $\alpha D^{-1}L + \frac{1}{\alpha} D^{-1}U = \begin{pmatrix} 0 & -1/10\alpha \\ \frac{-\alpha}{10} & 0 \end{pmatrix}$

$\therefore$ Char. poly $=$ $\lambda^2 - \left(\frac{-1}{10\cancel{\alpha}}\right)\left(\frac{-\cancel{\alpha}}{10}\right) = 0 \Rightarrow \lambda^2 - \frac{1}{100} = 0$

(independent of α)

$\therefore$ A is consistently ordered.

Also, M_J has only real eigenvalues and $\lambda_1 = \frac{1}{10}$, $\lambda_2 = -\frac{1}{10}$.

$\therefore$ SOR converges if $0 < \omega < 2$. $\quad \because \rho(M_J) < 1$

$\therefore$ Optimal $\omega_{opt} = \dfrac{2}{1 + \sqrt{1 - \rho(M_J)^2}} = \dfrac{2}{1 + \sqrt{1 - \left(\frac{1}{10}\right)^2}} = 1.0025126$

$\rho(M_{SOR,\omega_{opt}}) = \omega_{opt} - 1 = 0.0025126.$