Lecture 14:

Note: Previous example requires that $M = N^{-1}P$ is diagonalizable.

What if $N^{-1}P$ is NOT diagonalizable?

Theorem: Let $A \in M_{n\times n}(\mathbb{C})$ be a complex-valued matrix.

Then: $\lim_{k\to\infty} A^k = 0$ if and only if $\rho(A) < 1$.

Simple consequence:

Corollary: The iterative scheme $\vec{x}^{k+1} = M\vec{x}^k + \vec{b}$ converges if and only if $\rho(M) < 1$.

Proof: Consider

$$\begin{aligned} \vec{x}^{k+1} &= M\vec{x}^k + \vec{b} \\ \vec{x}^* &= M\vec{x}^* + \vec{b} \\ \hline \therefore\ \vec{e}^{k+1} &= M\vec{e}^k \Rightarrow \vec{e}^k = M^k\vec{e}^0 \end{aligned}$$

$\vec{x}^* =$ sol of $A\vec{x} = \vec{f}$.

$\vec{e}^k \to \vec{0}$ iff $M^k \to 0$ iff $\rho(M) < 1$.

Lecture 16:

Recall:

Theorem: Let $A \in M_{n\times n}(\mathbb{C})$ be a complex-valued matrix.

Then: $\lim_{k\to\infty} A^k = 0$ if and only if $\rho(A) < 1$.

Simple consequence:

Corollary: The iterative scheme $\vec{x}^{k+1} = M\vec{x}^k + \vec{b}$ converges if and only if $\rho(M) < 1$.

Proof of Theorem

($\Rightarrow$) Let λ be an eigenvalue of A with eigenvector $\vec{v}$.

Then: $A^k \vec{v} = \lambda^k \vec{v}$. Thus, $\underbrace{\lim_{k\to\infty} A^k}_{0} \vec{v} = \lim_{k\to\infty} \lambda^k \vec{v}$

$$\Rightarrow \quad \vec{0} = \underbrace{\left(\lim_{k\to\infty} \lambda^k\right)}_{0} \vec{v} \neq \vec{0}$$

$\therefore \quad |\lambda| < 1$ (for all eigenvalue λ)

$\therefore \quad \rho(A) < 1$

$$" \max_j \{ |\lambda_j| : \lambda_j \text{ is eigenvalue} \}$$

($\Leftarrow$) Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of A (can be repeated)

We apply the idea of Jordan Canonical Form

Find the invertible Q such that $Q^{-1}AQ$ becomes

Simple

ALMOST looks like a diagonal

matrix.

Useful Fact: (Can be used without proof)

For any $A \in M_{n\times n}(\mathbb{C})$, there exists an invertible $Q \in M_{n\times n}(\mathbb{C})$ such that $A = QJQ^{-1}$ where J is the Jordan Canonical Form of A. Actually,

$$J = \begin{pmatrix} J_{m_1}(\lambda_{i_1}) & & & O \\ & J_{m_2}(\lambda_{i_2}) & & \\ & & \ddots & \\ O & & & J_{m_s}(\lambda_{i_s}) \end{pmatrix}$$

(λ_{i_j} are eigenvalues of A)

where

$$\underbrace{J_{m_j}(\lambda_{i_j})} = \begin{pmatrix} \lambda_{i_j} & 1 & & O \\ & \lambda_{i_j} & \ddots & \\ & & \ddots & 1 \\ O & & & \lambda_{i_j} \end{pmatrix} \in M_{m_j \times m_j}(\mathbb{C})$$

$1 \le j \le s$

is called Jordan block with λ_{i_j}

e.g.

$$\bar{J} = \begin{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix} \end{pmatrix}$$

Now, $A^k = Q J^k Q^{-1}$ and

$$J^k = \begin{pmatrix} J_{m_1}^k(\lambda_{i_1}) & & \\ & \ddots & \\ & & J_{m_s}^k(\lambda_{i_s}) \end{pmatrix}$$

For $k \geq m_j - 1$

$$J_{m_j}^k(\lambda_{i_j}) = \begin{pmatrix} \lambda_{i_j}^k & C_1^k \lambda_{i_j}^{k-1} & C_2^k \lambda_{i_j}^{k-2} & \cdots & C_{m_j-1}^k \lambda_{i_j}^{k-m_j+1} \\ & \lambda_{i_j}^k & C_1^k \lambda_{i_j}^{k-1} & \cdots & C_{m_j-2}^k \lambda_{i_j}^{k-m_j+2} \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & C_1^k \lambda_{i_j}^{k-1} \\ & & & & \lambda_{i_j}^k \end{pmatrix} \quad \text{(By M.I.)}$$

$\because \rho(A) < 1 \quad \therefore \quad |\lambda_{i_j}| < 1$ for all j.

$$\therefore \lim_{k \to \infty} J_{m_j}^{k}(\lambda_{i_j}) = O \quad \text{for all } j$$

and $\displaystyle \lim_{k \to \infty} J^k = O$

$$\therefore \lim_{k \to \infty} A^k = \lim_{k \to \infty} Q J^k Q^{-1} = O$$

Splitting choice 3: Successive overrelaxation method (SOR)

Consider the iterative scheme:

(Suppose $A = L + D + U$)

$$L\vec{x}^{k+1} + D\vec{y}^{k+1} + U\vec{x}^{k} = \vec{b} \quad \text{——} \quad (*)$$

$$\vec{x}^{k+1} = \vec{x}^{k} + \underset{\mathbb{R}}{\omega}\left(\vec{y}^{k+1} - \vec{x}^{k}\right) \quad \text{——} \quad (**)$$

$$\Leftrightarrow \quad \vec{y}^{k+1} = \frac{1}{\omega}\left(\vec{x}^{k+1} + (\omega-1)\vec{x}^{k}\right)$$

Putting $(**)$ into $(*)$:

$$\left(L + \frac{1}{\omega}D\right)\vec{x}^{k+1} + \frac{1}{\omega}\left(\omega U + (\omega-1)D\right)\vec{x}^{k} = \vec{b}$$

or

$$\underbrace{\left(L + \frac{1}{\omega}D\right)}_{N}\vec{x}^{k+1} = \underbrace{\left(\frac{1}{\omega}D - (D+U)\right)}_{P}\vec{x}^{k} + \vec{b}$$

Remark: • SOR is equivalent to:

$$A = N - P = \underbrace{\left(L + \frac{1}{\omega}D\right)}_{N} - \underbrace{\left(\frac{1}{\omega}D - (D+U)\right)}_{P}$$

$$N\vec{x}^{k+1} = P\vec{x}^{k} + \vec{b} \iff \vec{x}^{k+1} = N^{-1}P\,\vec{x}^{k} + N^{-1}\vec{b}$$

• $$\rho(N^{-1}P) = \rho\Big(\underbrace{\left(L + \frac{1}{\omega}D\right)^{-1}\left(\frac{1}{\omega}D - (D+U)\right)}_{M_\omega}\Big)$$

has to be strictly less than 1 and it has to be small in order to converge fast.

• If $\omega = 1$, SOR = Gauss-Seidel