MATH 3310 Assignment 4

Due: April 9, 2021

1. Consider a $n \times n$ tridiagonal linear system $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{pmatrix} \alpha & -\gamma & & & \\ -\beta & \alpha & -\gamma & \dots & \\ & \ddots & \ddots & \ddots & \\ & & -\beta & \alpha & \end{pmatrix} \tag{1}$$

   where $\alpha \geq \beta$.

   (a) Suppose $\alpha = 4$, $\beta = 1$ and $\gamma = 4$. Prove that the Jacobi method to solve $A\mathbf{x} = \mathbf{b}$ converges by looking at the spectral radius of a suitable matrix. Please explain your answer with details.

   (b) Suppose $\alpha = 2$ $\beta = 1$ and $\gamma = 1$. Using the Housholder-John theorem, prove that the Gauss-Seidel method to solve $A\mathbf{x} = \mathbf{b}$ converges. Please explain your answer with details.

   (c) Suppose $\alpha = 4$, $\beta = 1$ and $\gamma = 4$. Explain why the SOR method converges for $0 < \omega < 2$. What is the optimal parameter ω_{opt} in the SOR method to obtain the fastest convergence. Please explain your answer with details.

2. Consider:

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

   Suppose an initial vector is given as $\mathbf{x}^{(0)} = (1, 1, 0)^T$. Calculate the first iteration of power method. Find the eigenvalue and the normalised eigenvector associated to it.

3. Consider:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{pmatrix}$$

   (a) Find the QR factorization of A.

   (b) Compute the first iteration of the QR iteration for A.

4. Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Let $x \in \mathbb{R}^{n\times 1}$ be an unit vector and

$$\alpha = \frac{x^T A x}{x^T x}, \text{ and } y = Ax - \alpha x$$

   (a) Let $\delta = \|y\|$. Prove that there exists an eigenvalue of A such that it is in the interval $[\alpha - \delta, \alpha + \delta]$.

(b) Suppose $\hat{x} = ax + by$. Show that there exists a unique pair of (a, b) up to some scaling(i.e. for all $c \in \mathbb{R}$, we consider (a, b) and (ca, cb) the same pair) such that the maximum of $\overline{\alpha} = \frac{\hat{x}^T A \hat{x}}{\hat{x}^T \hat{x}}$ is attained.

5. Suppose $A \in M_{n\times n}(\mathbb{C})$, with eigenvalues:

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \cdots > |\lambda_n| > 0$$

Also, we define the following:

$$\begin{aligned}
\cos \angle(x, y) &= \frac{|\langle x, y\rangle|}{\|x\|\|y\|};\\
\sin \angle(x, y) &= \sqrt{1 - \cos^2 \angle(x, y)};\\
\tan \angle(x, y) &= \frac{\sin \angle(x, y)}{\cos \angle(x, y)}.
\end{aligned}$$

We define $\tan \angle(x, y) = \infty$ if $\cos \angle(x, y) = 0$. Here, $\langle x, y\rangle = \sum_{i=1}^n x_i \bar{y}_i$, where x_i and y_i are the i-th entries of $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$ respectively.

(a) Suppose A is normal, i.e., $A = QDQ^*$ for some unitary Q (i.e. $Q^*Q = I$, where Q^* is the conjugate transpose of Q) and diagonal D, prove that

$$\cos \angle(Q^*x, Q^*y) = \cos \angle(x, y).$$

(b) Consider the power method in the form

$$x^{(n)} = Ax^{(n-1)}$$

Assume that $\cos \angle(x^{(0)}, e_1) \neq 0$, where e_1 is the eigenvector associated to the dominant eigenvalue λ_1. Prove that

$$\tan \angle(x^{(m+1)}, e_1) \leq \frac{|\lambda_2|}{|\lambda_1|} \tan \angle(x^{(m)}, e_1).$$

(c) For some $\mu \in \mathbb{R}$, let $A - \mu I$ be invertible. Assume

$$|\lambda_1 - \mu| < |\lambda_2 - \mu| \leq \cdots \leq |\lambda_n - \mu|.$$

Under the same notations and assumptions in (b), consider the shifted inverse power iteration

$$x^{(m)} = (A - \mu I)^{-1}x^{(m-1)}.$$

Using part (b) or otherwise, prove that

$$\tan \angle(x^{(m+1)}, e_1) \leq \frac{|\lambda_1 - \mu|}{|\lambda_2 - \mu|} \tan \angle(x^{(m)}, e_1).$$