MATH 3310 Assignment 3

Due: March 15, 2021

1. Consider the following linear system $Ax = b$, where

$$A = \begin{pmatrix} -3 & 3 & -6 \\ -4 & 7 & -8 \\ 5 & 7 & -9 \end{pmatrix} \quad and \quad b = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$$

   (a) Determine whether the Jacobi method converges.
   (b) Using initial approximation $x^0 = (1, 0, 0)^T$, conduct the first two Jacobi iterations.
   (c) Determine whether the Gauss-Seidel method converges.
   (d) Using initial approximation $x^0 = (0, 0, 1)^T$, conduct the first two Gauss-Seidel iterations.

2. Consider the following linear system $Ax = b$, where

$$A = \begin{pmatrix} 2 & -2 & -1 \\ -1 & 3 & -2 \\ 1 & -3 & 1 \end{pmatrix} \quad and \quad b = \begin{pmatrix} -1 \\ 7 \\ -7 \end{pmatrix}$$

   (a) Determine whether the SOR method converges if $\omega = 1.25$.
   (b) Using initial approximation $x^{(0)} = (0, 1, 0)^T$, conduct the first two SOR iteration.

3. Consider the following iterative scheme:

$$x_{k+1} = (2I - tA)x_k + tb$$

   Suppose that A is symmetric positive definite matrix in $\mathbb{R}^{n\times n}$, with eigenvalues $\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 > 0$.

   (a) Show that the above scheme converges if and only if $\frac{1}{\lambda_1} < t < \frac{3}{\lambda_n}$.
   (b) Prove that the optimal t, in the sense of rate of convergence, is $\frac{4}{\lambda_1 + \lambda_n}$

4. Consider the linear system $Ax = k$, where

$$A = \begin{pmatrix} 1 & 0 & -1/4 & -1/4 \\ 0 & 1 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1 & 0 \\ -1/4 & -1/4 & 0 & 1 \end{pmatrix} \quad and \quad k = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Let $x^* = (1,1,1,1)^T$ be the solution of the linear system. Suppose $\{x^{(m)}\}_{m=1}^{\infty}$ and $\{y^{(m)}\}_{m=1}^{\infty}$ are the sequences of vectors obtained by the Jacobi method and Gauss-Seidel method respectively to solve the linear system with initialization $x^{(0)} = y^{(0)} = (0,0,0,0)^T$. Let $e_J^{(m)} := x^{(m)} - x^*$ and $e_{GS}^{(m)} := y^{(m)} - x^*$ be the error vectors at the m-th iteration for the Jacobi and Gauss-Seidel method respectively.

(a) Show that: $e_J^{(m)} = -\frac{1}{2^m}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ for $m \geq 1$.

(b) Show that: $e_{GS}^{(m)} = -\frac{1}{4^m}\begin{pmatrix}2\\2\\1\\1\end{pmatrix}$ for $m \geq 1$.

(c) Show that $\|e_{GS}^{(m)}\|_2 < \|e_J^{(m)}\|_2$ for $m > 1$. Hence, the Gauss-Seidel method converges faster than the Jacobi method. Here, $\|(x_1, x_2, x_3, x_4)\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$.

5. Draw the butterfly diagram of the FFT F_8. (Hint: refer to the lecture notes for more details)

6. Let $a = (a_0, a_1, \cdots, a_{N-1})$ and $b = (b_0, b_1, \cdots, b_{N-1})$ be two vectors of length $N = 2^s$. In this question, you will write a Matlab code to compute the inner products of a with every cyclic shift of b. (for example: all the cyclic shifts of (1,2,3,4) are (1,2,3,4), (2,3,4,1), (3,4,1,2), and (4,1,2,3))

(a) What is the computational cost to compute all the inner products directly, i.e., compute $a \cdot \sigma(b)$ for every cyclic shift σ, *respectively.*

(b) Construct a new vector $\hat{b} = (b_{N-1}, b_{N-2}, \cdots, b_0)$. Let $c = a * \hat{b}$. Write down the formula for $c(m)$, where $m = 0, 1, \cdots, N-1$. Can you write some code to compute c using Matlab built-in function fft? Further, can you relate c with our given question? What is the computational cost of the new algorithm?