1. $y' + 3y = t + e^{-2t}$
2. $y' - 2y = t^2 e^{2t}$
3. $y' + y = te^{-t} + 1$
4. $y' + (1/t)y = 3\cos 2t, \qquad t > 0$
5. $y' - 2y = 3e^t$
6. $ty' + 2y = \sin t, \qquad t > 0$
7. $y' + 2ty = 2te^{-t^2}$
8. $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$
9. $2y' + y = 3t$
10. $ty' - y = t^2 e^{-t}, \qquad t > 0$
11. $y' + y = 5\sin 2t$
12. $2y' + y = 3t^2$

In each of Problems 13 through 20, find the solution of the given initial value problem.

13. $y' - y = 2te^{2t}, \qquad y(0) = 1$
14. $y' + 2y = te^{-2t}, \qquad y(1) = 0$
15. $ty' + 2y = t^2 - t + 1, \qquad y(1) = \frac{1}{2}, \quad t > 0$
16. $y' + (2/t)y = (\cos t)/t^2, \qquad y(\pi) = 0, \quad t > 0$
17. $y' - 2y = e^{2t}, \qquad y(0) = 2$
18. $ty' + 2y = \sin t, \qquad y(\pi/2) = 1, \quad t > 0$
19. $t^3 y' + 4t^2 y = e^{-t}, \qquad y(-1) = 0, \quad t < 0$
20. $ty' + (t + 1)y = t, \qquad y(\ln 2) = 1, \quad t > 0$

In each of Problems 21 through 23:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as t becomes large? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.

(b) Solve the initial value problem and find the critical value a_0 exactly.

(c) Describe the behavior of the solution corresponding to the initial value a_0.

21. $y' - \frac{1}{2}y = 2\cos t, \qquad y(0) = a$
22. $2y' - y = e^{t/3}, \qquad y(0) = a$
23. $3y' - 2y = e^{-\pi t/2}, \qquad y(0) = a$

In each of Problems 24 through 26:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as $t \to 0$? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.

(b) Solve the initial value problem and find the critical value a_0 exactly.

(c) Describe the behavior of the solution corresponding to the initial value a_0.

24. $ty' + (t + 1)y = 2te^{-t}, \qquad y(1) = a, \quad t > 0$
25. $ty' + 2y = (\sin t)/t, \qquad y(-\pi/2) = a, \quad t < 0$
26. $(\sin t)y' + (\cos t)y = e^t, \qquad y(1) = a, \quad 0 < t < \pi$
27. Consider the initial value problem

$$y' + \tfrac{1}{2}y = 2\cos t, \qquad y(0) = -1.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

28. Consider the initial value problem

$$y' + \tfrac{2}{3}y = 1 - \tfrac{1}{2}t, \qquad y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

29. Consider the initial value problem

$$y' + \tfrac{1}{4}y = 3 + 2\cos 2t, \qquad y(0) = 0.$$

(a) Find the solution of this initial value problem and describe its behavior for large t.
(b) Determine the value of t for which the solution first intersects the line $y = 12$.

30. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \qquad y(0) = y_0$$

remains finite as $t \to \infty$.

31. Consider the initial value problem

$$y' - \tfrac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

32. Show that all solutions of $2y' + ty = 2$ [Eq. (41) of the text] approach a limit as $t \to \infty$, and find the limiting value.
Hint: Consider the general solution, Eq. (47), and use L'Hôpital's rule on the first term.

33. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

In each of Problems 34 through 37, construct a first order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

34. All solutions have the limit 3 as $t \to \infty$.
35. All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.
36. All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.
37. All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$.
38. **Variation of Parameters.** Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \tag{i}$$

(a) If $g(t) = 0$ for all t, show that the solution is

$$y = A\exp\left[-\int p(t)\,dt\right], \tag{ii}$$

where A is a constant.
(b) If $g(t)$ is not everywhere zero, assume that the solution of Eq. (i) is of the form

$$y = A(t)\exp\left[-\int p(t)\,dt\right], \tag{iii}$$

then, by comparing numerators and denominators in Eqs. (26) and (27), we obtain the system

$$dx/dt = G(x, y), \qquad dy/dt = F(x, y). \tag{28}$$

At first sight it may seem unlikely that a problem will be simplified by replacing a single equation by a pair of equations, but in fact, the system (28) may well be more amenable to investigation than the single equation (27). Chapter 9 is devoted to nonlinear systems of the form (28).

Note 3: In Example 2 it was not difficult to solve explicitly for y as a function of x. However, this situation is exceptional, and often it will be better to leave the solution in implicit form, as in Examples 1 and 3. Thus, in the problems below and in other sections where nonlinear equations appear, the words "solve the following differential equation" mean to find the solution explicitly if it is convenient to do so, but otherwise to find an equation defining the solution implicitly.

PROBLEMS

In each of Problems 1 through 8, solve the given differential equation.

1. $y' = x^2/y$
2. $y' = x^2/y(1 + x^3)$
3. $y' + y^2 \sin x = 0$
4. $y' = (3x^2 - 1)/(3 + 2y)$
5. $y' = (\cos^2 x)(\cos^2 2y)$
6. $xy' = (1 - y^2)^{1/2}$
7. $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^y}$
8. $\dfrac{dy}{dx} = \dfrac{x^2}{1 + y^2}$

In each of Problems 9 through 20:

(a) Find the solution of the given initial value problem in explicit form.

(b) Plot the graph of the solution.

(c) Determine (at least approximately) the interval in which the solution is defined.

9. $y' = (1 - 2x)y^2, \qquad y(0) = -1/6$
10. $y' = (1 - 2x)/y, \qquad y(1) = -2$
11. $x\,dx + ye^{-x}dy = 0, \qquad y(0) = 1$
12. $dr/d\theta = r^2/\theta, \qquad r(1) = 2$
13. $y' = 2x/(y + x^2 y), \qquad y(0) = -2$
14. $y' = xy^3(1 + x^2)^{-1/2}, \qquad y(0) = 1$
15. $y' = 2x/(1 + 2y), \qquad y(2) = 0$
16. $y' = x(x^2 + 1)/4y^3, \qquad y(0) = -1/\sqrt{2}$
17. $y' = (3x^2 - e^x)/(2y - 5), \qquad y(0) = 1$
18. $y' = (e^{-x} - e^x)/(3 + 4y), \qquad y(0) = 1$
19. $\sin 2x\,dx + \cos 3y\,dy = 0, \qquad y(\pi/2) = \pi/3$
20. $y^2(1 - x^2)^{1/2}dy = \arcsin x\,dx, \qquad y(0) = 1$

Some of the results requested in Problems 21 through 28 can be obtained either by solving the given equations analytically or by plotting numerically generated approximations to the solutions. Try to form an opinion about the advantages and disadvantages of each approach.

21. Solve the initial value problem

$$y' = (1 + 3x^2)/(3y^2 - 6y), \qquad y(0) = 1$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

22. Solve the initial value problem

$$y' = 3x^2/(3y^2 - 4), \qquad y(1) = 0$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

23. Solve the initial value problem

$$y' = 2y^2 + xy^2, \qquad y(0) = 1$$

and determine where the solution attains its minimum value.

24. Solve the initial value problem

$$y' = (2 - e^x)/(3 + 2y), \qquad y(0) = 0$$

and determine where the solution attains its maximum value.

25. Solve the initial value problem

$$y' = 2\cos 2x/(3 + 2y), \qquad y(0) = -1$$

and determine where the solution attains its maximum value.

26. Solve the initial value problem

$$y' = 2(1 + x)(1 + y^2), \qquad y(0) = 0$$

and determine where the solution attains its minimum value.

27. Consider the initial value problem

$$y' = ty(4 - y)/3, \qquad y(0) = y_0.$$

(a) Determine how the behavior of the solution as t increases depends on the initial value y_0.
(b) Suppose that $y_0 = 0.5$. Find the time T at which the solution first reaches the value 3.98.

28. Consider the initial value problem

$$y' = ty(4 - y)/(1 + t), \qquad y(0) = y_0 > 0.$$

(a) Determine how the solution behaves as $t \to \infty$.
(b) If $y_0 = 2$, find the time T at which the solution first reaches the value 3.99.
(c) Find the range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

29. Solve the equation

$$\frac{dy}{dx} = \frac{ay + b}{cy + d},$$

where a, b, c, and d are constants.

Homogeneous Equations. If the right side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the ratio y/x only, then the equation is said to be homogeneous.[1] Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 30 illustrates how to solve first order homogeneous equations.

30. Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}. \tag{i}$$

(a) Show that Eq. (i) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}; \tag{ii}$$

thus Eq. (i) is homogeneous.

(b) Introduce a new dependent variable v so that $v = y/x$, or $y = xv(x)$. Express dy/dx in terms of x, v, and dv/dx.

(c) Replace y and dy/dx in Eq. (ii) by the expressions from part (b) that involve v and dv/dx. Show that the resulting differential equation is

$$v + x\frac{dv}{dx} = \frac{v - 4}{1 - v},$$

or

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \tag{iii}$$

Observe that Eq. (iii) is separable.

(d) Solve Eq. (iii), obtaining v implicitly in terms of x.

(e) Find the solution of Eq. (i) by replacing v by y/x in the solution in part (d).

(f) Draw a direction field and some integral curves for Eq. (i). Recall that the right side of Eq. (i) actually depends only on the ratio y/x. This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore, the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 30 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In each of Problems 31 through 38:

(a) Show that the given equation is homogeneous.

(b) Solve the differential equation.

(c) Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

31. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$

32. $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$

33. $\dfrac{dy}{dx} = \dfrac{4y - 3x}{2x - y}$

34. $\dfrac{dy}{dx} = -\dfrac{4x + 3y}{2x + y}$

[1] The word "homogeneous" has different meanings in different mathematical contexts. The homogeneous equations considered here have nothing to do with the homogeneous equations that will occur in Chapter 3 and elsewhere.

3. The possible points of discontinuity, or singularities, of the solution can be identified (without solving the problem) merely by finding the points of discontinuity of the coefficients. Thus, if the coefficients are continuous for all t, then the solution also exists and is differentiable for all t.

None of these statements is true, in general, of nonlinear equations. Although a nonlinear equation may well have a solution involving an arbitrary constant, there may also be other solutions. There is no general formula for solutions of nonlinear equations. If you are able to integrate a nonlinear equation, you are likely to obtain an equation defining solutions implicitly rather than explicitly. Finally, the singularities of solutions of nonlinear equations can usually be found only by solving the equation and examining the solution. It is likely that the singularities will depend on the initial condition as well as on the differential equation.

PROBLEMS

In each of Problems 1 through 6, determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1. $(t-3)y' + (\ln t)y = 2t, \qquad y(1) = 2$
2. $t(t-4)y' + y = 0, \qquad y(2) = 1$
3. $y' + (\tan t)y = \sin t, \qquad y(\pi) = 0$
4. $(4-t^2)y' + 2ty = 3t^2, \qquad y(-3) = 1$
5. $(4-t^2)y' + 2ty = 3t^2, \qquad y(1) = -3$
6. $(\ln t)y' + y = \cot t, \qquad y(2) = 3$

In each of Problems 7 through 12, state where in the ty-plane the hypotheses of Theorem 2.4.2 are satisfied.

7. $y' = \dfrac{t-y}{2t+5y}$
8. $y' = (1-t^2-y^2)^{1/2}$
9. $y' = \dfrac{\ln|ty|}{1-t^2+y^2}$
10. $y' = (t^2+y^2)^{3/2}$
11. $\dfrac{dy}{dt} = \dfrac{1+t^2}{3y-y^2}$
12. $\dfrac{dy}{dt} = \dfrac{(\cot t)y}{1+y}$

In each of Problems 13 through 16, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

13. $y' = -4t/y, \qquad y(0) = y_0$
14. $y' = 2ty^2, \qquad y(0) = y_0$
15. $y' + y^3 = 0, \qquad y(0) = y_0$
16. $y' = t^2/y(1+t^3), \qquad y(0) = y_0$

In each of Problems 17 through 20, draw a direction field and plot (or sketch) several solutions of the given differential equation. Describe how solutions appear to behave as t increases and how their behavior depends on the initial value y_0 when $t = 0$.

17. $y' = ty(3-y)$
18. $y' = y(3-ty)$
19. $y' = -y(3-ty)$
20. $y' = t-1-y^2$
21. Consider the initial value problem $y' = y^{1/3}$, $y(0) = 0$ from Example 3 in the text.
    (a) Is there a solution that passes through the point $(1, 1)$? If so, find it.
    (b) Is there a solution that passes through the point $(2, 1)$? If so, find it.
    (c) Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at $t = 2$.

22. (a) Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}, \qquad y(2) = -1.$$

Where are these solutions valid?

(b) Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.

(c) Show that $y = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$, the initial condition is also satisfied, and the solution $y = y_1(t)$ is obtained. Show that there is no choice of c that gives the second solution $y = y_2(t)$.

23. (a) Show that $\phi(t) = e^{2t}$ is a solution of $y' - 2y = 0$ and that $y = c\phi(t)$ is also a solution of this equation for any value of the constant c.

(b) Show that $\phi(t) = 1/t$ is a solution of $y' + y^2 = 0$ for $t > 0$ but that $y = c\phi(t)$ is not a solution of this equation unless $c = 0$ or $c = 1$. Note that the equation of part (b) is nonlinear, while that of part (a) is linear.

24. Show that if $y = \phi(t)$ is a solution of $y' + p(t)y = 0$, then $y = c\phi(t)$ is also a solution for any value of the constant c.

25. Let $y = y_1(t)$ be a solution of

$$y' + p(t)y = 0, \tag{i}$$

and let $y = y_2(t)$ be a solution of

$$y' + p(t)y = g(t). \tag{ii}$$

Show that $y = y_1(t) + y_2(t)$ is also a solution of Eq. (ii).

26. (a) Show that the solution (7) of the general linear equation (1) can be written in the form

$$y = cy_1(t) + y_2(t), \tag{i}$$

where c is an arbitrary constant.

(b) Show that y_1 is a solution of the differential equation

$$y' + p(t)y = 0, \tag{ii}$$

corresponding to $g(t) = 0$.

(c) Show that y_2 is a solution of the full linear equation (1). We see later (for example, in Section 3.5) that solutions of higher order linear equations have a pattern similar to Eq. (i).

Bernoulli Equations. Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form

$$y' + p(t)y = q(t)y^n,$$

and is called a Bernoulli equation after Jakob Bernoulli. Problems 27 through 31 deal with equations of this type.

27. (a) Solve Bernoulli's equation when $n = 0$; when $n = 1$.

(b) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method of solution was found by Leibniz in 1696.

In each of Problems 28 through 31, the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 27(b).

28. $t^2y' + 2ty - y^3 = 0, \qquad t > 0$
29. $y' = ry - ky^2$, $r > 0$ and $k > 0$. This equation is important in population dynamics and is discussed in detail in Section 2.5.
30. $y' = \epsilon y - \sigma y^3$, $\epsilon > 0$ and $\sigma > 0$. This equation occurs in the study of the stability of fluid flow.
31. $dy/dt = (\Gamma \cos t + T)y - y^3$, where Γ and T are constants. This equation also occurs in the study of the stability of fluid flow.

Discontinuous Coefficients. Linear differential equations sometimes occur in which one or both of the functions p and g have jump discontinuities. If t_0 is such a point of discontinuity, then it is necessary to solve the equation separately for $t < t_0$ and $t > t_0$. Afterward, the two solutions are matched so that y is continuous at t_0; this is accomplished by a proper choice of the arbitrary constants. The following two problems illustrate this situation. Note in each case that it is impossible also to make y' continuous at t_0.

32. Solve the initial value problem

$$y' + 2y = g(t), \qquad y(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & t > 1. \end{cases}$$

33. Solve the initial value problem

$$y' + p(t)y = 0, \qquad y(0) = 1,$$

where

$$p(t) = \begin{cases} 2, & 0 \le t \le 1, \\ 1, & t > 1. \end{cases}$$

2.5 Autonomous Equations and Population Dynamics

An important class of first order equations consists of those in which the independent variable does not appear explicitly. Such equations are called **autonomous** and have the form

$$dy/dt = f(y). \tag{1}$$

We will discuss these equations in the context of the growth or decline of the population of a given species, an important issue in fields ranging from medicine to ecology to global economics. A number of other applications are mentioned in some of the problems. Recall that in Sections 1.1 and 1.2 we considered the special case of Eq. (1) in which $f(y) = ay + b$.

Equation (1) is separable, so the discussion in Section 2.2 is applicable to it, but the main purpose of this section is to show how geometrical methods can be used to obtain important qualitative information directly from the differential equation without

(b) Show that $\int_0^1 2nxe^{-nx^2}\,dx = 1 - e^{-n}$; hence

$$\lim_{n\to\infty} \int_0^1 \phi_n(x)\,dx = 1.$$

Thus, in this example,

$$\lim_{n\to\infty} \int_a^b \phi_n(x)\,dx \neq \int_a^b \lim_{n\to\infty} \phi_n(x)\,dx,$$

even though $\lim_{n\to\infty} \phi_n(x)$ exists and is continuous.

In Problems 15 through 18, we indicate how to prove that the sequence $\{\phi_n(t)\}$, defined by Eqs. (4) through (7), converges.

15. If $\partial f/\partial y$ is continuous in the rectangle D, show that there is a positive constant K such that

$$|f(t, y_1) - f(t, y_2)| \le K|y_1 - y_2|, \tag{i}$$

where (t, y_1) and (t, y_2) are any two points in D having the same t coordinate. This inequality is known as a Lipschitz[20] condition.
Hint: Hold t fixed and use the mean value theorem on f as a function of y only. Choose K to be the maximum value of $|\partial f/\partial y|$ in D.

16. If $\phi_{n-1}(t)$ and $\phi_n(t)$ are members of the sequence $\{\phi_n(t)\}$, use the result of Problem 15 to show that

$$|f[t, \phi_n(t)] - f[t, \phi_{n-1}(t)]| \le K|\phi_n(t) - \phi_{n-1}(t)|.$$

17. (a) Show that if $|t| \le h$, then

$$|\phi_1(t)| \le M|t|,$$

where M is chosen so that $|f(t, y)| \le M$ for (t, y) in D.
(b) Use the results of Problem 16 and part (a) of Problem 17 to show that

$$|\phi_2(t) - \phi_1(t)| \le \frac{MK|t|^2}{2}.$$

(c) Show, by mathematical induction, that

$$|\phi_n(t) - \phi_{n-1}(t)| \le \frac{MK^{n-1}|t|^n}{n!} \le \frac{MK^{n-1}h^n}{n!}.$$

18. Note that

$$\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)].$$

(a) Show that

$$|\phi_n(t)| \le |\phi_1(t)| + |\phi_2(t) - \phi_1(t)| + \cdots + |\phi_n(t) - \phi_{n-1}(t)|.$$

(b) Use the results of Problem 17 to show that

$$|\phi_n(t)| \le \frac{M}{K}\left[Kh + \frac{(Kh)^2}{2!} + \cdots + \frac{(Kh)^n}{n!}\right].$$

(c) Show that the sum in part (b) converges as $n \to \infty$ and, hence, the sum in part (a) also converges as $n \to \infty$. Conclude therefore that the sequence $\{\phi_n(t)\}$ converges since it is the sequence of partial sums of a convergent infinite series.

[20]The German mathematician Rudolf Lipschitz (1832–1903), professor at the University of Bonn for many years, worked in several areas of mathematics. The inequality (i) can replace the hypothesis that $\partial f/\partial y$ is continuous in Theorem 2.8.1; this results in a slightly stronger theorem.

19. In this problem we deal with the question of uniqueness of the solution of the integral equation (3)

$$\phi(t) = \int_0^t f[s, \phi(s)]\, ds.$$

(a) Suppose that ϕ and ψ are two solutions of Eq. (3). Show that, for $t \geq 0$,

$$\phi(t) - \psi(t) = \int_0^t \{f[s, \phi(s)] - f[s, \psi(s)]\}\, ds.$$

(b) Show that

$$|\phi(t) - \psi(t)| \leq \int_0^t |f[s, \phi(s)] - f[s, \psi(s)]|\, ds.$$

(c) Use the result of Problem 15 to show that

$$|\phi(t) - \psi(t)| \leq K \int_0^t |\phi(s) - \psi(s)|\, ds,$$

where K is an upper bound for $|\partial f/\partial y|$ in D. This is the same as Eq. (30), and the rest of the proof may be constructed as indicated in the text.

2.9 First Order Difference Equations

Although a continuous model leading to a differential equation is reasonable and attractive for many problems, there are some cases in which a discrete model may be more natural. For instance, the continuous model of compound interest used in Section 2.3 is only an approximation to the actual discrete process. Similarly, sometimes population growth may be described more accurately by a discrete than by a continuous model. This is true, for example, of species whose generations do not overlap and that propagate at regular intervals, such as at particular times of the calendar year. Then the population y_{n+1} of the species in the year $n + 1$ is some function of n and the population y_n in the preceding year; that is,

$$y_{n+1} = f(n, y_n), \qquad n = 0, 1, 2, \ldots. \tag{1}$$

Equation (1) is called a **first order difference equation**. It is first order because the value of y_{n+1} depends on the value of y_n but not on earlier values y_{n-1}, y_{n-2}, and so forth. As for differential equations, the difference equation (1) is **linear** if f is a linear function of y_n; otherwise, it is **nonlinear**. A **solution** of the difference equation (1) is a sequence of numbers $y_0, y_1, y_2, \ldots$ that satisfy the equation for each n. In addition to the difference equation itself, there may also be an **initial condition**

$$y_0 = \alpha \tag{2}$$

that prescribes the value of the first term of the solution sequence.

We now assume temporarily that the function f in Eq. (1) depends only on y_n, but not on n. In this case

$$y_{n+1} = f(y_n), \qquad n = 0, 1, 2, \ldots. \tag{3}$$