MATH1520 University Mathematics for Applications **Spring 2021**

Chapter 3: Continuity

Learning Objectives:
(1) Explore the concept of continuity and examine the continuity of several functions.
(2) Investigate the intermediate value property.

3.1 Continuity

Definition 3.1.1. A function f is **continuous** at $x = x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$. It means all three of these conditions are satisfied:

1. $f(x_0)$ is defined.
2. $\lim_{x \to x_0} f(x)$ exists.
3. They are equal.

If some of (1)-(3) are not satisfied, then $f(x)$ is discontinuous at x_0.

If $f(x)$ is continuous at every point in the domain, $f(x)$ is called a continuous function.

Informally, a function $f(x)$ is continuous at $x = x_0$ if the curve of $f(x)$ does not break up at x_0. A continuous function is one whose graph has no holes or gaps.

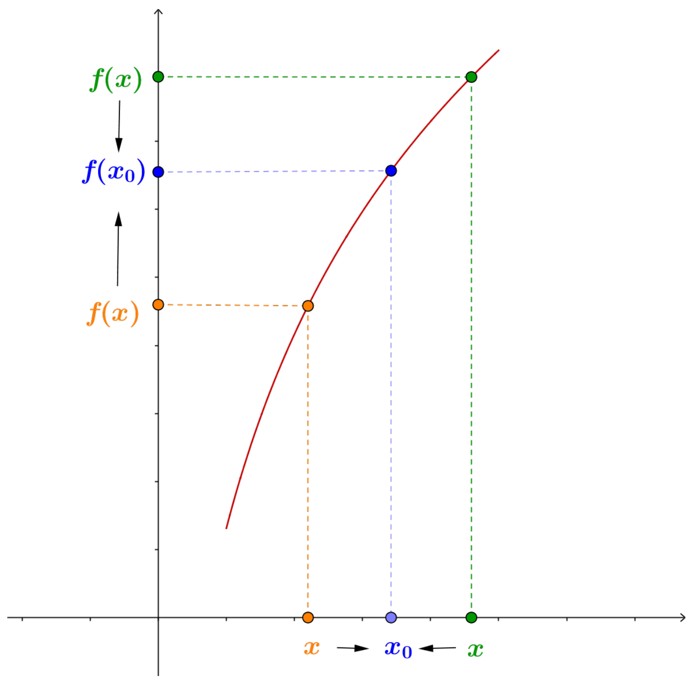
Example 3.1.1. Show that $f(x) = x^3 - 1$ is continuous at $x = 1$.

Solution.

$$f(1) = 0.$$

$$\lim_{x \to 1} f(x) = 1^3 - 1 = 0 = f(1)$$

(i.e., limit exists and is equal to $f(1)$.) ■



Example 3.1.2. Decide whether the function

$$f(x) = \begin{cases} x^3 - 1, & x \neq 1, \\ 1, & x = 1. \end{cases}$$

is continuous at $x = 1$.

Solution. Since

$$\lim_{x \to 1} f(x) = 0 \neq f(1),$$

$f(x)$ is discontinuous at $x = 1$. ■

Example 3.1.3. Discuss the continuity of $f(x) = \dfrac{1}{x}$.

Solution. $f(x)$ is defined everywhere except at $x = 0$, and $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}\ \forall c \neq 0$ by the first propositions of Chapter 2. So $f(x)$ is continuous for all $x \neq 0$. ■

Example 3.1.4. Piecewise linear functions (e.g. step functions, the ceil/floor function, $f(x) = |x|$); piecewise continuous functions.

Proposition 3.1.1. (Properties of continuity)

1. Suppose $f(x)$ and $g(x)$ are continuous at $x = x_0$. It follows from Proposition 2 in Chapter 2 that:
   (a) $f(x) + g(x)$, $f(x) - g(x)$, $f(x)g(x)$ are continuous at $x = x_0$.
   (b) If $g(x_0) \neq 0$, then $\dfrac{f(x)}{g(x)}$ is continuous at $x = x_0$.
2. It follows from Proposition 3 in Chapter 2 that: If $g(x)$ is continuous at $x = x_0$ and $f(x)$ is continuous at $x = g(x_0)$. Then $(f \circ g)(x)$, i.e., $f(g(x))$ is continuous at $x = x_0$. In fact $\lim_{x \to x_0} f(g(x)) = \lim_{u \to g(x_0)} f(u) = f(g(x_0))$.
3. $x^a, a^x, \log_a x$ and trig functions are all continuous functions in the domain. As a consequence, their $+, -, \times, \div, \circ$ are all continuous in the domain.

Example 3.1.5.

1. If $p(x)$ and $q(x)$ are polynomials, then
$$\lim_{x \to c} p(x) = p(c)$$
and
$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \text{ if } q(c) \neq 0.$$
So a polynomial or a rational function is continuous wherever it is defined (i.e. $q(c) \neq 0$).
2. $f(x) = \dfrac{x-1}{x+1}$ is continuous at $x = 2$.
3. $f(x) = \dfrac{x^2 - 1}{x+1}$ is defined everywhere except at $x = -1$, so it is continuous everywhere except at $x \neq -1$.
4. $g(x) = \ln \sqrt{x^2 + 1}$ is continuous on $\mathbb{R}$.

Example 3.1.6. Discuss the continuity of the piecewise function:

$$f(x) = \begin{cases} x+1 & \text{if } x \le 1, \\ 2x^2 & \text{if } x > 1. \end{cases}$$

Solution. For $x < 1$, $f(x) = x + 1$ is continuous on $(-\infty, 1)$;

For $x > 1$, $f(x) = 2x^2$ is continuous on $(1, +\infty)$;

At $x = 1$, $f(1) = 1 + 1 = 2$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+1) = 1 + 1 = 2.$$

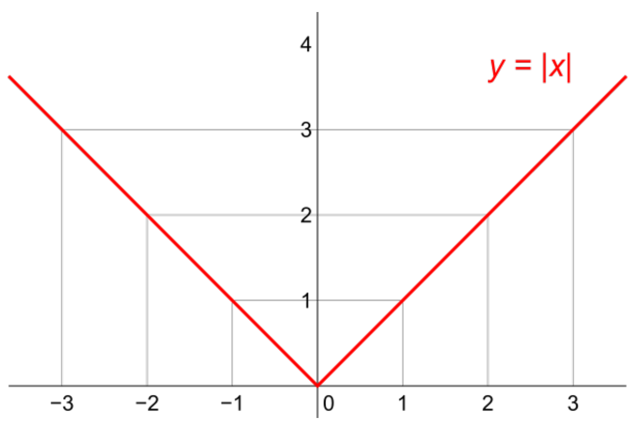
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x^2 = 2 \cdot 1^2 = 2.$$

Because the left hand limit and the right hand limit exist and equal. So $\lim_{x \to 1} f(x) = 2 = f(1)$. Therefore $f(x)$ is continuous at all x. ■

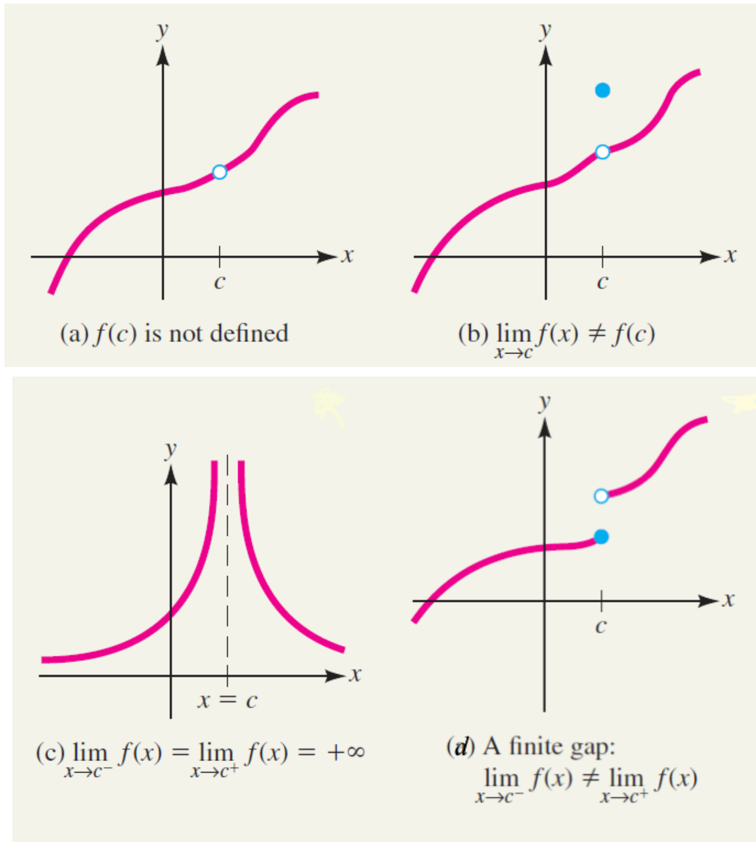
Example 3.1.7.

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

$|x|$ is a continuous everywhere and $\lim_{x \to 0} |x| = 0$.



Example 3.1.8. (Discontinuity)



(a) $f(c)$ is not defined

(b) $\lim_{x\to c} f(x) \neq f(c)$

(c) $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = +\infty$

(d) A finite gap: $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$

Example 3.1.9. For what value of A such that the following function is continuous at all x?

$$f(x) = \begin{cases} x^2 + x - 1 & \text{if } x \leq 0, \\ x + A & \text{if } x > 0. \end{cases}$$

Solution. Because $x^2 + x - 1$ and $x + A$ are polynomials, they are continuous everywhere except possibly at $x = 0$. Also $f(0) = 0^2 + 0 - 1 = -1$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2 + x - 1) = -1$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x + A) = A.$$

For $\lim_{x\to 0} f(x)$ to exist, the left hand limit and the right hand limit must be equal. So we must have $A = -1$. In which case

$$\lim_{x\to 0} f(x) = -1 = f(0).$$

This means that $f(x)$ is continuous for all x only when $A = -1$. ■

Proposition 3.1.2. $f(x)$ is continuous at $x = c$ if and only if

$$\lim_{h\to 0} f(c+h) = f(c).$$

Proof. Let $h = x - c$. Then $h \to 0$ as $x \to c$.

$$\lim_{x\to c} f(x) = \lim_{h\to 0} f(c+h).$$

□

Exercise 3.1.1.

1. Show that $\sqrt[3]{x^3+1}$ is a continuous function.
2. Show that $\left|\dfrac{x+1}{x-1}\right|$ is a continuous function on $\mathbb{R}\backslash\{1\}$.
3. Let

$$f(x) = \begin{cases} x^2-1, & x \le 0, \\ x+a, & x > 0. \end{cases}$$

   Find a such that $f(x)$ is continuous at 0. (Ans: $a = -1$)

Example 3.1.10 (Using continuity to compute limits). $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = ?$

3.2 Continuity on $[a, b]$

Definition 3.2.1. Let $f : (a, b) \to \mathbb{R}$ be a function. Then f is said to be continuous on (a, b) if it is continuous at every point on (a, b).

Next, let's assume $f : [a, b] \to \mathbb{R}$ be a function. What's the meaning of f being continuous at one of the end point a? $\lim_{x\to a} f(x)$ does not make sense because f is not defined on $x < a$. So to define the continuity at $x = a$, we only concern about the value $x > a$. Similarly, to discuss about the continuity at $x = b$, we only concern about the value $x < b$.

Definition 3.2.2. Let $f : [a, b] \to \mathbb{R}$ be a function.
Then f is said to be continuous at a if

$$\lim_{x\to a^+} f(x) = f(a).$$

f is said to be continuous at b if

$$\lim_{x\to b^-} f(x) = f(b).$$

Then f is said to be a continuous function on $[a, b]$ if f is continuous on $a \le x \le b$.

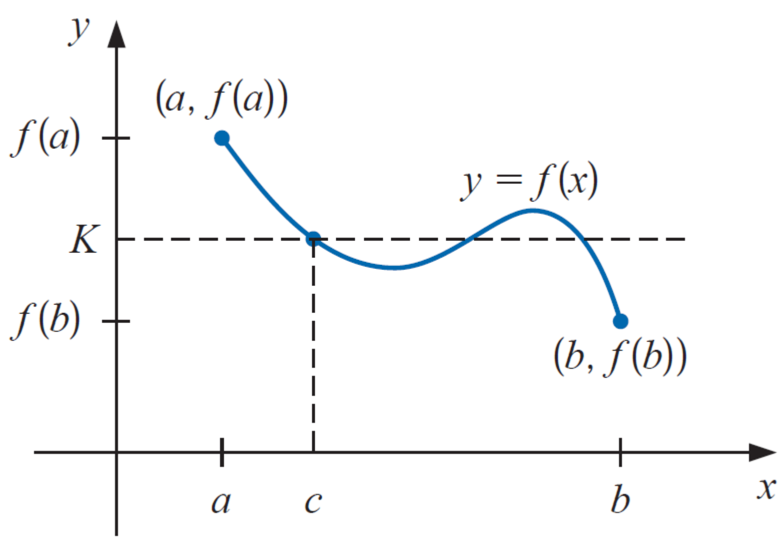
Example 3.2.1. Discuss the continuity of the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{x-1}{x} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$$

Solution. $f(x)$ is continuous on $(0,1)$. $f(x)$ is also continuous at $x = 1$, but $\lim_{x \to 0^+} f(x)$ does not exists. So f is not continuous at $x = 0$. ■

Theorem 3.2.1 (Intermediate Value Theorem or Intermediate Value Property). *Suppose f is a continuous function on $[a,b]$ and K is a number between $f(a)$ and $f(b)$. Then there exist a number c, between a and b, such that $f(c) = K$.*

Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a,b]$.



Application: Root Finding

If $f(x)$ is continuous on $[a,b]$, $f(a)$ and $f(b)$ change sign, then, there exists at least one root of the function, that is, exists at least one $c \in (a,b)$, such that $f(c) = 0$.

Example 3.2.2. Show that $f(x) = x^5 - x + 1$ has a root.

Solution. Aim: find a, b, such that $f(a)$, $f(b)$ change sign. Since

$$f(-2) = -29, \quad f(0) = 1,$$

and f is continuous on $[-2, 0]$. By Intermediate value theorem, there exists $c \in (-2, 0)$, such that $f(c) = 0$.

■

Remark. Although we don't know how to find the root, we know a root exists.

Example 3.2.3. 1. All odd functions have a root.

2. All polynomials of odd degrees have a root.

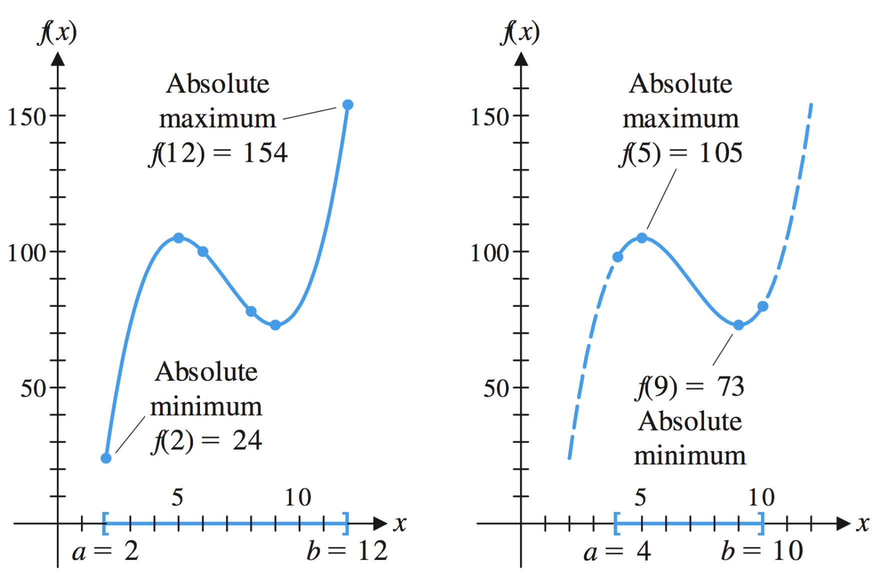
Exercise 3.2.1. Show that $2^x = \frac{1}{x^2}$ has a solution.

Theorem 3.2.2 (Extreme Value Theorem)**.** *If $f(x)$ is continuous on $[a, b]$, then f must attain an absolute maximum and absolute minimum, that is, there exist c, d in $[a, b]$ such that*

$$f(c) \leq f(x) \leq f(d),$$

for all $x \in [a, b]$.

Example 3.2.4. Absolute extreme for $f(x) = x^3 - 21x^2 + 135x - 170$ for various closed intervals.



(A) [a, b] = [2, 12] (B) [a, b] = [4, 10]

Exercise 3.2.2 (Hard!). Derive the extreme value theorem from the intermediate value theorem.

Remark. Caveat: The intermediate value theorem and the extreme value theorem only work on *finite* and *closed* intervals! E.g. Consider the previous example on $\mathbb{R}$, and $\frac{1}{x}$ on $\mathbb{R}^+$ or on $(0,1)$.

Question: How to find the absolute maximum and minimum?

Ans: (for "good" functions) Differentiation!