Solution to assignment 9

(1) (16.3, Q28):
$\frac{\partial P}{\partial y} = \cos z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{e^x}{y} = \frac{\partial M}{\partial y}$
$\Rightarrow \mathbf{F}$ is conservative.
$\Rightarrow$ There exists a f so that $\mathbf{F} = \nabla f$.
$\frac{\partial f}{\partial x} = e^x \ln y$
$\Rightarrow f(x, y, z) = e^x \ln y + g(y, z)$
$\Rightarrow \frac{\partial f}{\partial y} = \frac{e^x}{y} + \frac{\partial g}{\partial y} = \frac{e^x}{y} + \sin z$
$\Rightarrow \frac{\partial g}{\partial y} = \sin z$
$\Rightarrow g(y, z) = y \sin z + h(z)$
$\Rightarrow f(x, y, z) = e^x \ln y + y \sin z + h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = y \cos z + h'(z) = y \cos z$
$\Rightarrow h'(z) = 0$
$\Rightarrow h(z) = C$
$\Rightarrow f(x, y, z) = e^x \ln y + y \sin z + C$
$\Rightarrow \mathbf{F} = \nabla \left(e^x \ln y + y \sin z\right).$

(2) (16.4, Q14):
$M = \tan^{-1} \frac{y}{x}, N = \ln\left(x^2 + y^2\right).$
$\Rightarrow \frac{\partial M}{\partial x} = \frac{-y}{x^2+y^2}, \frac{\partial M}{\partial y} = \frac{x}{x^2+y^2}, \frac{\partial N}{\partial x} = \frac{2x}{x^2+y^2}, \frac{\partial N}{\partial y} = \frac{2y}{x^2+y^2}$
$\Rightarrow$ Flux $= \iint_R \left(\frac{-y}{x^2+y^2} + \frac{2y}{x^2+y^2}\right) dxdy = \int_0^\pi \int_1^2 \left(\frac{r \sin \theta}{r^2}\right) r dr d\theta = \int_0^\pi \sin \theta d\theta = 2$
Circ $= \iint_0 \left(\frac{2x}{x^2+y^2} - \frac{x}{x^2+y^2}\right) dxdy = \int_0^\pi \int_1^2 \left(\frac{r \cos \theta}{r^2}\right) r dr d\theta = \int_0^\pi \cos \theta d\theta = 0.$

(3) (16.4, Q27):
$M = x = \cos^3 t, N = y = \sin^3 t$
$\Rightarrow dx = -3 \cos^2 t \sin t dt, dy = 3 \sin^2 t \cos t dt$
$\Rightarrow$ Area $= \frac{1}{2} \oint_C x dy - y dx$
$= \frac{1}{2} \int_0^{2\pi} \left(3 \sin^2 t \cos^2 t\right) \left(\cos^2 t + \sin^2 t\right) dt$
$= \frac{1}{2} \int_0^{2\pi} \left(3 \sin^2 t \cos^2 t\right) dt$
$= \frac{3}{8} \int_0^{2\pi} \sin^2 2t dt$
$= \frac{3}{16} \int_0^{4\pi} \sin^2 u du$
$= \frac{3}{16} \left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi}$
$= \frac{3}{8}\pi.$

(4) (16.4, Q39):
(a) $\nabla f = \left(\frac{2x}{x^2+y^2}\right) \mathbf{i} + \left(\frac{2y}{x^2+y^2}\right) \mathbf{j} \Rightarrow M = \frac{2x}{x^2+y^2}, N = \frac{2y}{x^2+y^2}.$

M, N are discontinuous at $(0,0)$, we compute $\int_C \nabla f \cdot \mathbf{n} ds$ directly since Green's Theorem does not apply.
Let $x = a\cos t, y = a \sin t$
$\Rightarrow dx = -a \sin t dt, dy = a \cos t dt$
$\Rightarrow M = \frac{2}{a}\cos t, N = \frac{2}{a}\sin t, 0 \le t \le 2\pi$.
So $\int_C \nabla f \cdot \mathbf{n} ds = \int_C M dy - N dx$
$= \int_0^{2x} \left[\left(\frac{2}{a}\cos t\right)(a\cos t) - \left(\frac{2}{a}\sin t\right)(-a\sin t)\right] dt$
$= \int_0^{2\pi} 2\left(\cos^2 t + \sin^2 t\right) dt$
$= 4\pi$.
Note that this holds for any $a > 0$, so $\int_C \nabla f \cdot \mathbf{n} ds = 4\pi$ for any circle C centered at (0,0) traversed counterclockwise and $\int_C \nabla f \cdot \mathbf{n} ds = -4\pi$ if C is traversed clockwise.
(b) If K does not enclose the point (0,0) we may apply Green's Theorem:
$\int_C \nabla f \cdot \mathbf{n} ds = \int_C M dy - N dx$
$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy$
$= \iint_R \left(\frac{2\left(y^2 - x^2\right)}{\left(x^2+y^2\right)^2} + \frac{2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right) dx dy$
$= \iint_R 0 dx dy = 0$.
If K does enclose the point $(0,0)$, we proceed as follows:
Choose a small enough so that the circle C centered at (0,0) of radius a lies entirely within K. Green's Theorem applies to the region R that lies between K and C.
Thus, as before, $0 = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy = \int_K M dy - N dx + \int_C M dy - N dx$ where K is traversed counterclockwise and C is traversed clockwise.
Hence by part (a),

$$0 = \int_K M dy - N dx - 4\pi,$$

$$\int_K \nabla f \cdot \mathbf{n} ds = \int_K M dy - N dx = 4\pi.$$

We have shown that $\int_K \nabla f \cdot \mathbf{n} ds = \begin{cases} 0 & \text{if } (0,0) \text{ lies inside K,} \\ 4\pi & \text{if } (0,0) \text{ lies outside K.} \end{cases}$

Solution to Assignment 9

Supplementary Problems

1. A vector field $\mathbf{F}$ is called radial if $\mathbf{F}(x,y,z) = f(r)(x,y,z),\ r = |(x,y,z)|$, for some function f. Show that every radial vector field is conservative. You may assume it is C^1 in $\mathbb{R}^3$.

   Solution. Let $\Phi(x,y,z)$ be the potential. Since f is radially symmetric, we believe that Φ is also radially symmetric. Let $\Phi(x,y,z) = \varphi(r),\ r = \sqrt{x^2+y^2+z^2}$. We have

   $$\frac{\partial \Phi}{\partial x} = \varphi'(r)\frac{x}{r}, \quad \frac{\partial \Phi}{\partial y} = \varphi'(r)\frac{y}{r}, \quad \frac{\partial \Phi}{\partial z} = \varphi'(r)\frac{z}{r} \ .$$

   By comparison, we see that Φ is a potential for $\mathbf{F}$ if $\varphi'(r)/r = f(r)$. Therefore,

   $$\varphi(r) = \int_0^r t f(t)\, dt \ ,$$

   is a potential for $\mathbf{F}$.

2. Let $F = (P,Q)$ be a C^1-vector field in $\mathbb{R}^2$ away from the origin. Suppose that $P_y = Q_x$. Show that for any simple closed curve C enclosing the origin and oriented in positive direction, one has

   $$\oint_C Pdx + Qdy = \lim_{\varepsilon\to 0} \varepsilon \int_0^{2\pi} \left[-P(\varepsilon\cos\theta, \varepsilon\sin\theta)\sin\theta + Q(\varepsilon\cos\theta,\varepsilon\sin\theta)\cos\theta\right] d\theta \ .$$

   What happens when C does not enclose the origin?

3. We identity the complex plane with $\mathbb{R}^2$ by $x+iy \mapsto (x,y)$. A complex-valued function f has its real and imaginary parts respectively given by $u(x,y) = Ref(z)$ and $v(x,y) = Imf(z)$. Note that u and v are real-valued functions. The function f is called differentiable at z if

   $$\frac{df}{dz}(z) = \lim_{w\to 0} \frac{f(z+w)-f(z)}{w} \ ,$$

   exists.

   (a) Show that f is differentiable at z implies that the partial derivatives of u and v exist and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, hold. Hint: Take $w = h, ih$, where $h \in \mathbb{R}$ and then let $h \to 0$.

   Solution. Identify z with (x,y). As f is differentiable at z, for real h,

   $$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\left(\frac{u(x+h,y)-u(x,y)}{h} + i\frac{v(x+h,y)-v(x,y)}{h}\right) \\ &= \lim_{h\to 0}\frac{u(x+h,y)-u(x,y)}{h} + i\lim_{h\to 0}\frac{v(x+h,y)-v(x,y)}{h} \ . \end{aligned}$$

   Using the fact that $a_n + ib_n \to a + ib$ if and only if $a_n \to a$ and $b_n \to b$ (here $f'(z) = a+ib$), we see that $\partial u/\partial x$ and $\partial v/\partial x$ exists and

   $$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}(x,y) = f'(z) \ .$$

Next, we consider purely imaginary ih,

$$\begin{aligned} f'(z) &= \lim_{h\to 0}\frac{f(z+h)-f(z)}{ih} = \lim_{h\to 0}\left(-i\frac{u(x,y+h)-u(x,y)}{h} + \frac{v(x,y+h)-v(x,y)}{h}\right) \\ &= -i\lim_{h\to 0}\frac{u(x,y+h)-u(x,y)}{h} + \lim_{h\to 0}\frac{v(x,y+h)-v(x,y)}{h} \ . \end{aligned}$$

As before, $\partial u/\partial y$ and $\partial v/\partial y$ exists and

$$-i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) = f'(z) \ .$$

By comparison, we have $\partial v/\partial y = \partial u/\partial x$ and $-\partial u/\partial y = \partial v/\partial x$ at (x, y).

(b) Propose a definition of $\int_C f\,dz$, where C is an oriented curve in the plane, in terms of the line integrals involving u and v.

Solution. Formally we have $fdz = (u + iv)(dx + idy) = udx - vdy + i(vdx + udy)$. So, we define

$$\int_C f\,dz = \int_C udx - vdy + i\int_C vdx + udy \ .$$

Note that the right hand side are two line integrals.

(c) Suppose that f is differentiable everywhere in $\mathbb{C}$. Show that for every simple closed curve C,

$$\oint_C f\,dz = 0 \ .$$

Solution. Use (a) we see that $P = u, Q = -v$ as well as $P = v, Q = u$ satisfy the compatibility conditions. Hence, by Green's theorem,

$$\oint_C f\,dz = 0 \ .$$

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The conclusion in (c) is called Cauchy's theorem. It is a fundamental result in complex analysis.