Solution to assignment 12

(1) (16.8, Q17):

(a)

$$\frac{\partial}{\partial x}(x) = 1, \frac{\partial}{\partial y}(y) = 1, \frac{\partial}{\partial z}(z) = 1$$

$$\nabla \cdot \mathbf{F} = 3$$

$$\text{Flux} = \iiint_D 3dV = 3\iiint_D dV = 3(\text{Volume of the solid})$$

(b) If $\mathbf{F}$ is orthogonal to $\mathbf{n}$ at every point of S, then $\mathbf{F} \cdot \mathbf{n} = 0$ everywhere.
$\Rightarrow$ Flux $= \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = 0$.
But the flux is 3 (Volume of the solid) $\neq 0$, so $\mathbf{F}$ is not orthogonal to $\mathbf{n}$ at every point.

(2) (16.8, Q19):

$$\mathbf{F} = (y\cos 2x)\mathbf{i} + \left(y^2 \sin 2x\right)\mathbf{j} + \left(x^2 y + z\right)\mathbf{k}$$

$$\nabla \cdot \mathbf{F} = -2y\sin 2x + 2y \sin 2x + 1 = 1.$$

If $\mathbf{F}$ is the curl of a field $\mathbf{A}$ whose component functions have continuous second partial derivatives, then we would have

$$\operatorname{div} \mathbf{F} = \operatorname{div}(\operatorname{curl} \mathbf{A}) = \nabla \cdot (\nabla \times \mathbf{A}) = 0.$$

Since $\operatorname{div} \mathbf{F} = 1$, $\mathbf{F}$ is not the curl of such a field.

(3) (16.8, Q20):

From the Divergence Theorem,

$$\iint_S \nabla f \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \nabla f dV = \iiint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) dV.$$

Now we have

$$f(x,y,z) = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln\left(x^2+y^2+z^2\right)$$

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2+z^2}, \frac{\partial f}{\partial y} = \frac{y}{x^2+y^2+z^2}, \frac{\partial f}{\partial z} = \frac{z}{x^2+y^2+z^2}.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2}, \frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2+z^2}{\left(x^2+y^2+z^2\right)^2}, \frac{\partial^2 f}{\partial z^2} = \frac{x^2+y^2-z^2}{\left(x^2+y^2+z^2\right)^2}.$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2} = \frac{1}{x^2+y^2+z^2}.$$

Thus we have
$\iint_S \nabla f \cdot \mathbf{n} d\sigma$
$= \iiint_D \frac{dV}{x^2+y^2+z^2}$
$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \frac{\rho^2 \sin\phi}{\rho^2} d\rho d\phi d\theta$
$= \int_0^{\pi/2} \int_0^{\pi/2} a \sin\phi d\phi d\theta$
$= \int_0^{\pi/2} [-a\cos\phi]_0^{\pi/2} d\theta$
$= \int_0^{\pi/2} a d\theta$
$= \dfrac{\pi a}{2}.$

(4) (16.8, Q21):
The integral's value never exceeds the surface area of S.
Since $|\mathbf{F}| \leq 1$, we have

$$|\mathbf{F} \cdot \mathbf{n}| = |\mathbf{F}||\mathbf{n}| \leq 1$$

Then we have
$\iiint_D \nabla \cdot \mathbf{F} d\sigma$
$= \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma$
$\leq \iint_S |\mathbf{F} \cdot \mathbf{n}| d\sigma$
$\leq \iint_S 1 d\sigma$
$=$ Area of S.