Solution to assignment 11

(1) (16.7, Q6):

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 3x^2y^2\mathbf{k}$$

$$\mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{4}$$

$$\operatorname{curl}\mathbf{F} \cdot \mathbf{n} = -\frac{3}{4}x^2y^2z$$

$$d\sigma = \frac{4}{z}dA \quad (\text{ Section 16.6, Example 6, with } a = 4)$$

$\oint_C \mathbf{F} \cdot d\mathbf{r}$
$= \iint_R \left(-\frac{3}{4}x^2y^2z\right)\left(\frac{4}{z}\right) dA$
$= -3\int_0^{2\pi}\int_0^2 (r^2\cos^2\theta)\left(r^2\sin^2\theta\right) rdrd\theta$
$= -3\int_0^{2\pi} \left[\frac{r^6}{6}\right]_0^2 (\cos\theta\sin\theta)^2 d\theta$
$= -32\int_0^{2\pi} \frac{1}{4}\sin^2 2\theta d\theta$
$= -4\int_0^{4\pi} \sin^2 u du$
$= -4\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi}$
$= -8\pi.$

(2) (16.7, Q15):

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix} = -2y^3\mathbf{i} + 0\mathbf{j} - x^2\mathbf{k}$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k}$$

$$\nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, drd\theta$$

$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$
$= \iint_R (2ry^3\cos\theta - rx^2)\, drd\theta$
$= \int_0^{2\pi}\int_0^1 \left(2r^4\sin^3\theta\cos\theta - r^3\cos^2\theta\right) drd\theta$
$= \int_0^{2\pi} \left(\frac{2}{5}\sin^3\theta\cos\theta - \frac{1}{4}\cos^2\theta\right) d\theta$
$= \left[\frac{1}{10}\sin^4\theta - \frac{1}{4}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\right]_0^{2\pi}$

$= -\frac{\pi}{4}$.

(3) (16.8, Q14):
Let $\rho = \sqrt{x^2+y^2+z^2}$, then we have

$$\frac{\partial\rho}{\partial x} = \frac{x}{\rho}, \frac{\partial\rho}{\partial y} = \frac{y}{\rho}, \frac{\partial\rho}{\partial z} = \frac{z}{\rho}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{\rho}\right) = \frac{1}{\rho} - \left(\frac{x}{\rho^2}\right)\frac{\partial\rho}{\partial x} = \frac{1}{\rho} - \frac{x^2}{\rho^3}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{\rho}\right) = \frac{1}{\rho} - \frac{y^2}{\rho^3}$$

$$\frac{\partial}{\partial z}\left(\frac{y}{\rho}\right) = \frac{1}{\rho} - \frac{z^2}{\rho^3}$$

$$\nabla\cdot\mathbf{F} = \frac{3}{\rho} - \frac{x^2+y^2+z^2}{\rho^3} = \frac{2}{\rho}$$

Flux $= \iiint_D \frac{2}{\rho} dV$
$= \int_0^{2\pi}\int_0^{\pi}\int_1^2 \left(\frac{2}{\rho}\right)(\rho^2\sin\phi)\, d\rho d\phi d\theta$
$= \int_0^{2\pi}\int_0^{\pi} 3\sin\phi d\phi d\theta$
$= \int_0^{2\pi} 6d\theta$
$= 12\pi$.

(4) (16.8, Q16):

$$\frac{\partial}{\partial x}\left[\ln\left(x^2+y^2\right)\right] = \frac{2x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left(-\frac{2z}{x}\tan^{-1}\frac{y}{x}\right) = \left(-\frac{2z}{x}\right)\left[\frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2}\right] = -\frac{2z}{x^2+y^2}$$

$$\frac{\partial}{\partial z}\left(z\sqrt{x^2+y^2}\right) = \sqrt{x^2+y^2}$$

$$\nabla\cdot\mathbf{F} = \frac{2x}{x^2+y^2} - \frac{2z}{x^2+y^2} + \sqrt{x^2+y^2}$$

Flux $= \iiint_D \left(\frac{2x}{x^2+y^2} - \frac{2z}{x^2+y^2} + \sqrt{x^2+y^2}\right) dzdydx$
$= \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-1}^2 \left(\frac{2r\cos\theta}{r^2} - \frac{2z}{r^2} + r\right) dzrdrd\theta$
$= \int_0^{2\pi}\int_1^{\sqrt{2}} \left(6\cos\theta - \frac{3}{r} + 3r^2\right) drd\theta$
$= \int_0^{2\pi}[6(\sqrt{2}-1)\cos\theta - 3\ln\sqrt{2} + 2\sqrt{2} - 1]d\theta$

$$= 2\pi \left(-\tfrac{3}{2}\ln 2 + 2\sqrt{2} - 1\right).$$