Interchange of Limits

Uniform convergence is essential when we want to interchange the order of limits. The first proposition tells us that **continuity** is preserved under uniform convergence.

Theorem (c.f. Theorem 8.2.2)**.** *Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ and converges uniformly to a function f defined on A. Suppose that each f_n is continuous on A. Then f is continuous on A.*

Remark. This theorem tells us that for each $x_0 \in A$,

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{x\to x_0} f(x) = f(x_0) = \lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x).$$

Example 1. The assumption on uniform convergence cannot be dropped. Consider

$$f_n(x) = x^n, \quad x \in [0,1], \quad n \in \mathbb{N}.$$

The pointwise limit of (f_n) is given by $f(x) = 0$ for $x \in [0,1)$ and $f(1) = 1$. However, the convergence on $[0,1]$ is not uniform. For, choose $n_k = k$ and $x_k = (0.5)^{1/k} \in [0,1]$. Then

$$|f_{n_k}(x_k) - f(x_k)| = \left|[(0.5)^{1/k}]^k - 0\right| = \frac{1}{2} > 0.$$

In this case, each f_n is continuous on $[0,1]$ but f is not.

The second proposition tells us that **Riemann integrability** is preserved under uniform convergence.

Theorem (c.f. Theorem 8.2.4)**.** *Let (f_n) be a sequence of functions defined on $[a,b]$ and converges uniformly to a function f defined on $[a,b]$. Suppose that $f_n \in \mathcal{R}[a,b]$ for all $n \in \mathbb{N}$. Then $f \in \mathcal{R}[a,b]$ and*

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Remark. Does the same result hold for improper integrals?

Example 2 (Section 8.2, Ex.16)**.** The assumption on uniform convergence cannot be dropped. Let (r_n) be an enumeration of all rational numbers in $[0,1]$. Consider

$$f_n(x) = \begin{cases} 1, & \text{if } x \in \{r_1, r_2, ..., r_n\}, \\ 0, & \text{otherwise,} \end{cases} \qquad x \in [0,1], \quad n \in \mathbb{N}.$$

- If $x \in [0,1] \cap \mathbb{Q}$, then $x = r_N$ for some N and hence $f_n(x) = 1$ for all $n \geq N$.
- If $x \in [0,1] \setminus \mathbb{Q}$, then $f_n(x) = 0$ for all $n \in \mathbb{N}$.

Hence the pointwise limit of (f_n) is given by the **Dirichlet's function** f. i.e., $f(x) = 1$ for $x \in [0,1] \cap \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{Q}\setminus[0,1]$. However, the convergence on $[0,1]$ is not uniform. For, choose $n_k = k$ and $x_k \in [0,1]$ be a rational number that does not belongs to $\{r_1, r_2, ..., r_k\}$. Then

$$|f_{n_k}(x_k) - f(x_k)| = |0 - 1| = 1 > 0.$$

In this case, each f_n is Riemann integrable over $[0,1]$, but f is not.

Example 3. Even if the pointwise limit is Riemann integrable, the equality given in the theorem may not hold. Consider (Draw the graphs of the functions!)

$$f_n(x) = \begin{cases} n^2x, & \text{if } 0 \leq x < 1/n, \\ -n^2x + 2n, & \text{if } 1/n \leq x < 2/n, \\ 0, & \text{if } 2/n \leq x \leq 1, \end{cases} \quad x \in [0,1], \quad n \geq 2.$$

- If $x = 0$, then $f_n(x) = n^2x = 0$ for all $n \geq 2$.
- If $x \in (0,1]$, then $2/N \leq x$ for some $N \geq 2$ and hence $f_n(x) = 0$ for all $n \geq N$.

Hence the pointwise limit of (f_n) is given by the **zero function** f. However, the convergence on $[0,1]$ is not uniform. For, choose $n_k = k$ and $x_k = 1/k \in [0,1]$. Then

$$|f_{n_k}(x_k) - f(x_k)| = |k - 0| = k \geq 1 > 0.$$

In this case, each f_n is Riemann integrable over $[0,1]$ and so does f. However,

$$\int_0^1 f(x)dx = 0 \quad \text{and} \quad \int_0^1 f_n(x)dx = \frac{1}{2} \cdot \frac{2}{n} \cdot n = 1, \quad \forall n \geq 2.$$

Therefore

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx.$$

The preservation of differentiability is a bit different. Conditions on the derivatives of the sequences of functions are emphasised.

Theorem (c.f. Proposition 4.1 of Lecture Note)**.** *Let (f_n) be a sequence of $\mathcal{C}^1$ functions defined on (a,b) and converges to a function f defined on (a,b). Suppose (f_n') converges uniformly to a function g on (a,b). Then*

- *f is a $\mathcal{C}^1$ function on (a,b); and*
- *$f' = g$ on (a,b).*

Theorem (c.f. Proposition 4.2 of Lecture Note)**.** *Let (f_n) be a sequence of differentiable functions defined on (a,b). Suppose that there exists a point $c \in (a,b)$ such that $\lim_{n\to\infty} f_n(c)$ exists and (f_n') converges uniformly to a function g on (a,b). Then*

- *(f_n) converges uniformly to a differentiable function f on (a,b); and*
- *$f' = g$ on (a,b).*

Example 4. Even if we have uniform convergence of (f_n), the assumption on uniform convergence of (f_n') cannot be dropped. Consider (Draw the graphs of the functions!)

$$f_n(x) = \begin{cases} |x|, & \text{if } |x| > 1/n, \\ (n^2x^2+1)/2n, & \text{if } |x| \le 1/n, \end{cases} \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Notice that (f_n) converges uniformly to the absolute value function $f(x) = |x|$, which is not differentiable at 0. To see the uniform convergence, we have:

- If $|x| \le 1/n$, then
$$\Big|f_n(x) - |x|\Big| = \left|\frac{n^2x^2+1}{2n} - |x|\right| = \frac{(n|x|-1)^2}{2n} \le \frac{1}{2n}.$$

- If $|x| > 1/n$, then
$$\Big|f_n(x) - |x|\Big| = \Big||x| - |x|\Big| = 0 \le \frac{1}{2n}.$$

Thus given any $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that $1/N < 2\varepsilon$. Then

$$\Big|f_n(x) - |x|\Big| \le \frac{1}{2n} < \frac{1}{2N} < \varepsilon, \quad \forall n \ge N, \quad \forall x \in \mathbb{R}.$$

In this case, each f_n is $\mathcal{C}^1$ on $\mathbb{R}$, with derivative given by

$$f_n'(x) = \begin{cases} -1, & \text{if } x < -1/n, \\ nx, & \text{if } |x| \le 1/n, \\ 1, & \text{if } x > 1/n, \end{cases} \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

- If $x = 0$, then $f_n'(x) = nx = 0$ for all $n \in \mathbb{N}$.
- If $x < 0$, then $-x < 1/N$ for some N and hence $f_n'(x) = -1$ for all $n \ge N$.
- If $x > 0$, then $x < 1/N$ for some N and hence $f_n'(x) = 1$ for all $n \ge N$.

Hence the pointwise limit of (f_n') is given by the **sign function** sgn. However, the convergence on $\mathbb{R}$ is not uniform because each f_n' is continuous but the limit sgn is not.

Example 5 (c.f. Section 8.2, Ex.4)**.** Suppose (f_n) is a sequence of continuous functions defined on an interval I that converges uniformly to a function f on I. If $(x_n) \subseteq I$ converges to $x_0 \in I$, show that

$$\lim_{n\to\infty} f_n(x_n) = f(x_0).$$

Solution. We need to show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x_n) - f(x_0)| < \varepsilon, \quad \forall n \ge N.$$

Notice that for any $n \in \mathbb{N}$,

$$|f_n(x_n) - f(x_0)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|.$$

Let $\varepsilon > 0$. Since (f_n) converges to f uniformly, there exists $N_1 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \quad \forall n \geq N_1, \quad \forall x \in I.$$

On the other hand, note that f is a continuous function because it is the uniform limit of a sequence of continuous functions. In particular, $\lim f(x_n) = f(x_0)$. i.e., there exists $N_2 \in \mathbb{N}$ such that

$$|f(x_n) - f(x_0)| < \frac{\varepsilon}{2}, \quad \forall n \geq N_2.$$

Combining the above results, take $N = \max\{N_1, N_2\}$. Then whenever $n \geq N$,

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Remark. In the very beginning, we estimate $|f_n(x_n) - f(x_0)|$ by

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|.$$

What happens if we change the estimation to

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f_n(x_0)| + |f_n(x_0) - f(x_0)|?$$