The L'Hospital's Rule

L'Hospital's Rule (c.f. 6.3.3 & 6.3.5)**.** *Let $-\infty \leq a < b \leq \infty$ and lef f, g be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that*

$$\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x) \quad \text{or} \quad \lim_{x\to a^+} g(x) = \pm\infty.$$

If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$.

Remark. A similar argument shows that this theorem still works if we consider the left-hand limit $x \to a^-$ instead of the right-hand limit $x \to a^+$. We can combine these two versions and get the version for usual limit. (See **Example 1** below).

Example 1. Evaluate $\lim_{x\to 0} \frac{\arctan x}{x}$.

Solution. Let $f(x) = \arctan x$ and $g(x) = x$. Note that the derivatives of f and g are

$$f'(x) = \frac{1}{1+x^2} \quad \text{and} \quad g'(x) = 1, \quad \forall x \in \mathbb{R}.$$

In particular, consider the two functions f and g on $(0, 1)$. Notice that:

- Both f and g are differentiable on $(0, 1)$ and $g'(x) \neq 0$ for all $x \in (0, 1)$.
- $\lim_{x\to 0^+} f(x) = \arctan 0 = 0$ and $\lim_{x\to 0^+} g(x) = 0$.
- $\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{1}{1+x^2} = \frac{1}{1+0^2} = 1.$

Hence by the **The L'Hospital's Rule**, the right-hand limit is computed by:

$$\lim_{x\to 0^+} \frac{\arctan x}{x} = \lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = 1.$$

Similarly, considering the functions defined on $(-1, 0)$ gives the corresponding results for left-hand limit. It follows that

$$\lim_{x\to 0} \frac{\arctan x}{x} = \lim_{x\to 0^+} \frac{\arctan x}{x} = \lim_{x\to 0^-} \frac{\arctan x}{x} = 1.$$

Example 2. Evaluate $\lim_{x\to 0^+} x^x$.

Solution. Notice that we have the following expression for x^x:

$$x^x = e^{x \ln x} = \exp(x \ln x), \quad \forall x > 0.$$

Since the exponential function is continuous, we have

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} \exp(x \ln x) = \exp\left(\lim_{x\to 0^+} x \ln x\right) = \exp\left(\lim_{x\to 0^+} \frac{\ln x}{1/x}\right).$$

Let $f(x) = \ln x$ and $g(x) = 1/x$. Note that the derivatives of f and g are

$$f'(x) = \frac{1}{x} \quad \text{and} \quad g'(x) = -\frac{1}{x^2}, \quad \forall x > 0.$$

In particular, consider the two functions f and g on $(0, 1)$. Notice that:

- Both f and g are differentiable on $(0, 1)$ and $g'(x) \neq 0$ for all $x \in (0, 1)$.
- $\lim_{x\to 0^+} g(x) = \infty$.
- $\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} -x = 0$.

Hence by the **L'Hospital's Rule**, the limit is computed by

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = 0.$$

Hence required limit is

$$\lim_{x\to 0^+} x^x = \exp\left(\lim_{x\to 0^+} \frac{\ln x}{1/x}\right) = \exp(0) = e^0 = 1.$$

Theorem (c.f. Proposition 1.21 of Lecture Note)**.** *Let f be a function defined on (a, b) and c be a point in the interval (a, b).*

(a) If f is differentiable at c, then

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h}.$$

(b) If f is differentiable on (a, b) and $f''(c)$ exists, then

$$f''(c) = \lim_{h\to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2}.$$

Proof. For (a), notice that we cannot apply the **L'Hospital's Rule** because the function $F(h) = f(c + h) - f(x - h)$ may not be differentiable on a neighbourhood of 0. Indeed, notice that for sufficiently small $h \neq 0$, (so that $c \pm h \in (a, b)$)

$$\frac{f(c+h) - f(c-h)}{2h} = \frac{f(c+h) - f(c)}{2h} + \frac{f(c) - f(c-h)}{2h}.$$

Hence using the fact that f is differentiable at c,

$$\begin{aligned} \lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h} &= \frac{1}{2} \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} + \frac{1}{2} \lim_{h\to 0} \frac{f(c-h) - f(c)}{-h} \\ &= \frac{1}{2} f'(c) + \frac{1}{2} f'(c) = f'(c) \end{aligned}$$

For (b), consider the functions F and G defined by

$$F(h) = f(c+h) + f(c-h) - 2f(c) \quad \text{and} \quad G(h) = h^2, \quad \forall h \in (-\delta, \delta).$$

Here, $\delta > 0$ is chosen small enough such that $a < c - \delta < c + \delta < b$. We have

$$F'(h) = f'(c+h) - f'(c-h) \quad \text{and} \quad G'(h) = 2h, \quad \forall h \in (-\delta, \delta).$$

In this case, notice that:

- Both F and G are differentiable on $(-\delta, \delta)$ and $G'(h) \neq 0$ for all $h \in (-\delta, \delta) \setminus \{0\}$.
- Since f is differentiable on (a, b), it is continuous on (a, b). Hence

$$\lim_{h\to 0} F(h) = f(c+0) + f(c-0) - 2f(c) = 0 \quad \text{and} \quad \lim_{h\to 0} G(h) = 0^2 = 0.$$

- Since f' is differentiable at c. Using (a), we have

$$\lim_{h\to 0} \frac{F'(h)}{G'(h)} = \lim_{h\to 0} \frac{f'(c+h) - f'(c-h)}{2h} = f''(c).$$

Hence be applying the **L'Hospital's Rule**,

$$\lim_{h\to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2} = \lim_{h\to 0} \frac{F(h)}{G(h)} = \lim_{h\to 0} \frac{F'(h)}{G'(h)} = f''(c).$$

The result follows. □

We will then present the proof of the L'Hospital's Rule in the remaining section.

Proof of the L'Hospital's Rule. The case for $g(x) \to \infty$ as $x \to a^+$ will be presented. The remaining case is left as an exercise. Let $\varepsilon > 0$. We need to show that there exists some $d \in (a, b)$ such that

$$\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon \quad \iff \quad L - \varepsilon < \frac{f(x)}{g(x)} < L + \varepsilon, \quad \forall x \in (a, d).$$

Since $f'(x)/g'(x) \to L$ as $x \to a^+$, there exists $c \in (a, b)$ such that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \frac{\varepsilon}{2}, \quad \forall x \in (a, c). \tag{1}$$

As $g(x) \to \infty$ as $x \to a^+$, we can further assume that $g(x) > 0$ for all $x \in (a, c]$ by "pushing" c closer to a. If we fix any $x \in (a, c)$ and apply the **Cauchy Mean Value Theorem** on the interval $[x, c]$, we have

$$\frac{f(c) - f(x)}{g(c) - g(x)} = \frac{f'(u)}{g'(u)}, \quad \text{for some } u \in (x, c) \subseteq (a, c).$$

Putting into (1), the following holds for any $x \in (a, c)$:

$$\left|\frac{f(c)-f(x)}{g(c)-g(x)} - L\right| < \frac{\varepsilon}{2} \quad\iff\quad L - \frac{\varepsilon}{2} < \frac{f(c)-f(x)}{g(c)-g(x)} < L + \frac{\varepsilon}{2} \tag{2}$$

On the other hand, note that $g(c)/g(x) \to 0$ as $x \to a^+$. There exists $c' \in (a, c)$ such that

$$0 < \frac{g(c)}{g(x)} = \left|\frac{g(c)}{g(x)} - 0\right| < 1 \quad\implies\quad 0 < \frac{g(x)-g(c)}{g(x)} < 1, \quad \forall x \in (a, c').$$

Multiplying the above fraction to (2), the following holds for any $x \in (a, c')$:

$$\left(L - \frac{\varepsilon}{2}\right)\left(1 - \frac{g(c)}{g(x)}\right) < \frac{f(x)-f(c)}{g(x)} < \left(L + \frac{\varepsilon}{2}\right)\left(1 - \frac{g(c)}{g(x)}\right) \tag{3}$$

Let $0 < \delta \le 1$. Using the fact that $g(c)/g(x) \to 0$ and $f(c)/g(x) \to 0$ as $x \to a^+$, there exists $d \in (a, c')$ such that

$$0 < \frac{g(c)}{g(x)} < \delta, \quad \text{and} \quad -\delta < \frac{f(c)}{g(x)} < \delta, \quad \forall x \in (a, d).$$

Hence putting into (3) gives:

$$\left(L - \frac{\varepsilon}{2}\right)(1-\delta) - \delta < \frac{f(x)}{g(x)} < \left(L + \frac{\varepsilon}{2}\right) + \delta, \quad \forall x \in (a, d). \tag{4}$$

If $\delta \le \varepsilon/2$, then the right-hand inequality of (4) becomes $f(x)/g(x) < L + \varepsilon$, which is as desired. It suffices to take a suitable $0 < \delta \le \varepsilon/2$ such that left-hand inequality of (4) is not less than $L - \varepsilon$. Note that

$$\begin{aligned}
\left(L - \frac{\varepsilon}{2}\right)(1-\delta) - \delta \ge L - \varepsilon \quad &\iff\quad L - \frac{\varepsilon}{2} - \delta(1+L) + \frac{\varepsilon\delta}{2} \ge L - \varepsilon \\
&\iff\quad \frac{\varepsilon}{2} + \frac{\varepsilon\delta}{2} \ge \delta(1+L) \\
&\iff\quad \frac{1+\delta}{\delta} \ge \frac{2(1+L)}{\varepsilon} \\
&\iff\quad \frac{1}{\delta} \ge \frac{2(1+L)}{\varepsilon} - 1
\end{aligned}$$

This can be achieved by taking $\delta \in (0, 1]$ small enough, so the result follows. □