MATH 2060B - HW 2 - Solutions[1]

1 (P.179 Q5)**.** Let $a > b > 0$ and let $n \in \mathbb{N}$ satisfy $n \geq 2$. Show that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$.
Hint: Consider the function $f(x) := x^{1/n} - (x-1)^{1/n}$. Show that f is decreasing for $x \geq 1$ and evaluate f at 1 and a/b

Solution.
Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = x^{1/n} - (x-1)^{1/n}$. We proceed to show that $f'(x) < 0$ for all $x \in (1, \infty)$. By differentiation formula for power functions, linearity of differentiation and the chain rule, we can compute that

$$f'(x) = \frac{1}{n}x^{(1-n)/n} - \frac{1}{n}(x-1)^{(1-n)/n} = \frac{1}{n}(x^{(1-n)/n} - (x-1)^{(1-n)/n})$$

for all $x \in (1, \infty)$. Since $n \geq 2$, $\frac{1-n}{n} < 0$. For all $x \in (1, \infty)$, with the fact that $x > x - 1 \geq 0$, we have $x^{(1-n)/n} < (x-1)^{(1-n)/n}$ (since the exponent $\frac{1-n}{n}$ is a rational number, the inequality can be proved readily from the algebraic and order axioms for real numbers). Hence, $f'(x) = \frac{1}{n}(x^{(1-n)/n} - (x-1)^{(1-n)/n}) < 0$ for all $x \in (1, \infty)$.
Now there are two ways to arrive at the next check point: $f(a/b) < f(1)$

Method 1: Using the Mean Value Theorem (MVT) directly. Since $a > b$, we have $a/b > 1$. Note that f is continuous on $[0, a/b]$ and differentiable on $(0, a/b)$. Hence by MVT, there exists $\xi \in (0, a/b)$ such that $f(a/b) - f(1) = f'(\xi)(a/b - 1)$. From what have been proven, we have $f(a/b) - f(1) < 0$ as $f'(\xi) < 0$.

Method 2: Interpreting the monotonicity of f from its derivatives We first need to establish the following fact.

Proposition 0.1. *Let I be an open interval and $f : I \to \mathbb{R}$ be a function differentiable on I. Suppose $f'(x) \leq 0$ (resp. $f'(x) < 0$) for all $x \in I$. Then f is decreasing (resp. strictly decreasing) on I.*

Proof. Please refer to Theorem 6.2.7 of the textbook and the discussion afterwards or just try to prove yourself. The result in fact follows almost readily from the Mean Value Theorem. □

As a result f is strictly decreasing on $(1, \infty)$. Next, we would like to pass to strict monotoncity to include the endpoint 1 to show that $f(x) < f(1)$ for all $x \in (1, \infty)$. Fix $x \in (1, \infty)$. Choose a *decreasing* sequence (x_n) in $(1, x)$ such that $x_n \to 1$. Then we have $f(x) < f(x_n)$ for all $n \in \mathbb{N}$ by the strict decrease of f. Since f is continuous on $[1, \infty)$, we have $f(1) = \lim_n f(x_n)$ and so $(f(x_n))$ is a bounded sequence (why?). Since f is strictly decreasing and (x_n) is decreasing, $(f(x_n))$ is an increasing sequence. By the Bounded Monotone Convergence Theorem, we have $f(1) = \lim_n f(x_n) = \sup_n f(x_n)$. We then have $f(x) < f(x_N) \leq f(1)$ upon choosing some $x_N \in \{x_n\}$. As a result $f(a/b) < f(1)$ as $a/b > 1$.

Finally, the result follows by noting that

$$f(\frac{a}{b}) < f(1) \iff a^{1/n} - b^{1/n} < (a-b)^{1/n}$$

Comment.

1. The *strictly* in *strictly decreasing* is crucial. By now, you should learn that the difference between strict inequalities ($<$) and inequalities ($\leq$) can be huge. For example, the converse of Proposition 0.1 in Method 2 is true for the non-strict case but in general not true for the strict case.

2. Also in Method 2, an argument passing to the case of *endpoints* is crucial. Even though there may be only a few points, it can induce large problems. For example, the Interior Extremum Theorem does not apply to endpoints.

3. We give an alternative solution to the problem without using the hint on next page.

[1] Please feel free to email your TA at kllam@math.cuhk.edu.hk for any questions concerning homework.

Alternative Solution to Question 1. Since $a > b > 0$, we have $0 < (a-b)^{1/n} < b^{1/n} + (a-b)^{1/n}$ and $0 < b^{1/n} < b^{1/n} + (a-b)^{1/n}$. Hence, we have

$$0 < \frac{(a-b)^{1/n}}{b^{1/n} + (a-b)^{1/n}} < 1 \qquad 0 < \frac{b^{1/n}}{b^{1/n} + (a-b)^{1/n}} < 1$$

Since $n \geq 2$, for all $x \in (0,1)$, we have $0 < x^n < x$ (why?). Hence by taking nth power, we have

$$0 < \frac{a-b}{(b^{1/n} + (a-b)^{1/n})^n} < \frac{(a-b)^{1/n}}{b^{1/n} + (a-b)^{1/n}} \tag{1}$$

$$0 < \frac{b}{(b^{1/n} + (a-b)^{1/n})^n} < \frac{b^{1/n}}{b^{1/n} + (a-b)^{1/n}} \tag{2}$$

By summing up the inequalities, we have

$$\begin{aligned} & \frac{a-b}{(b^{1/n} + (a-b)^{1/n})^n} + \frac{b}{(b^{1/n} + (a-b)^{1/n})^n} < \frac{(a-b)^{1/n}}{b^{1/n} + (a-b)^{1/n}} + \frac{b^{1/n}}{b^{1/n} + (a-b)^{1/n}} \\ \iff & \frac{a}{(b^{1/n} + (a-b)^{1/n})^n} < 1 \\ \iff & a^{1/n} - b^{1/n} < (a-b)^{1/n} \end{aligned}$$

Remark. The inequality in question is in fact equivalent to proving that for all $n \geq 2, n \in \mathbb{N}, x, y > 0$, we have

$$x + y < (x^{1/n} + y^{1/n})^n$$

In fact, the inequality remains true as long as $n > 1$ even if $n \in \mathbb{R}$. In addition, the quantity of the right-hand side is called the **1/n norm** of the point $(x, y) \in \mathbb{R}^2$, denoted by $\|(x,y)\|_{1/n}$.

In general, we have $\|(x,y)\|_p := (|x|^p + |y|^p)^{1/p}$ for all $(x,y) \in \mathbb{R}^2$. as long as $p > 0$, which is called the **p-norm** of (x, y). In fact for all $(x, y) \in \mathbb{R}$ and $p > q > 0$, we always have

$$\|(x,y)\|_p := (|x|^p + |y|^p)^{1/p} \leq \|(x,y)\|_q := (|x|^q + |y|^q)^{1/q}$$

The inequality in question concerns the case where $p = 1$ and $q = 1/n$ where $n \in \mathbb{N}$. (Note that the last inequality is not strict as we include all $(x, y) \in \mathbb{R}^2$)

2 (P.179 Q14)**.** Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I.
Suppose f' is never 0 on I. Show that either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.

Solution. Suppose not. There exists x, y such that $f'(x) < 0, f'(y) > 0$. We can proceed in two way.

Method 1: Using Compactness. Suppose $x < y$. Since f is differentiable on I, f is continuous on I. Hence, f is continuous on $[x, y]$, which is closed and bounded, and so a compact set. By Extreme Value Theorem, f attains minimum at $x_* \in [x, y]$. We proceed to show that x_* in fact lies in (x, y) and hence in the interior of I.
Suppose it were true that $x_* = x$. Since $f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} < 0$, there exists $r > 0$ such that $\frac{f(t)-f(x)}{t-x} < 0$ for all $t \in I \cap (x - r, x + r)\backslash\{x\}$. In particular when $x < t$ and $|x - t| < r$, we have $f(t) < f(x)$, that is f is locally strict decreasing at x. As $x_* = x$, the minimality of x_* is contradicted. Similarly, f is locally strictly increasing at y so it is impossible that $x_* = y$. To conclude , we must have $x_* \in (x, y)$, so it lies in an open interval on which f is differentiable. By Proposition 1.10 (the Interior Extremum Theorem), an interior local extreme point has vanishing derivative, so $f'(x_*) = 0$, which is a contradiction to the assumption.
The case for $x > y$ could be done similarly by considering a local maximum. The result follows.

Method 2: Using Connectedness. Without loss of generality, suppose $x < y$.
Case 1: suppose $f(x) = f(y)$. Note that f is continuous on $[x, y]$ and is differentiable on (x, y) by assumption. Hence by Rolle's Theorem. There exists $\xi \in (x, y)$ such that $f(\xi) = 0$. Contradiction arises.
Case 2: suppose $f(x) < f(y)$. Since $f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} > 0$, there exists $r > 0$ such that $\frac{f(t)-f(x)}{t-x} < 0$ for all $t \in I \cap (x - r, x + r)\backslash\{x\}$. In particular, there exists $z \in (x, y)$ such that $f(z) < f(x)$. Therefore $f(z) < f(x) < f(y)$. Hence, by the Intermediate Value Theorem, as f is continuous on $[z, y]$, there exists $\xi \in (z, y)$ such that $f(\xi) = f(x)$. Contradiction arises by using the Rolle's Theorem to ξ, x.
Case 3: suppose $f(y) < f(x)$. Consider the $f'(y) > 0$ instead of $f'(x) < 0$ as in Case 2.
Therefore, in any case, contradiction arises and the result follows.

Comment.

1. Using Darboux's Theorem (Theorem 6.2.12 in the textbook) is not accepted. This is because basically, the statement here *is* the Darboux's Theorem with some special points (with $k = 0$). It is just pointless to use this Theorem in this question (I believe you knew that when you were doing this question). You should be able to develop a sense on when to prove Theorems in details or you risk having mark deduction in assessments (or in the other extreme, having not enough time because too much is written). A rule of thumb is to stick to the Lecture and Tutorial Notes.

2. In method 1, to use the Interior Extremum Theorem, one must check the the global extremum induced by the Extremem Value Theorem does not attain at the boundaries. To do so, one should use the locally strict monotonicity at the points instead of simplying stating that the derivatives there are nonzero. This is basically how the Darboux's Theorem is proved in the textbook.

3. In method 2, use the Intermediate Value Theorem on f, NOT the derivative f'. In general, derivatives may not be continuous in the case of real numbers.

4. In line with the Lecture notes, we shall only consider the case where I is an open interval. Otherwise, one may find it confusing if the x, y in the above proof are endpoints in which case their derivatives have not been defined. Nonetheless, the solution above still follows if we are using differentiability at boundaries as defined in Definition 6.1.1 in the textbook (using the same difference quotient but with 1-sided limit).

3 (P.196 Q4)**.** Let $x > 0$. Show that

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$$

Solution. Let $f : [-1, \infty)$ be defined by $f(t) = \sqrt{1+t}$. Now fix $x > 0$. Note that f is smooth (infinitely differentiable) on $(-1, \infty)$ (which is clear from the chain rule and the differentiability of power functions) and $0, x \in (-1, \infty)$. In particular, f, f' are continuous on $[0, x]$ and f'' exists on $(0, x)$. By Taylor's Theorem on f, there exists $\xi \in (0, x)$ such that

$$f(x) - f(0) = f'(0)(x-0) + \frac{1}{2!}f''(\xi)(x-0)^2$$

In the above, $f''(\xi) = \frac{-1}{4}(1+\xi)^{-3/2}$. Hence, we have $\frac{-1}{4} < f''(\xi) < 0$. This is because $\xi > 0$ and so we have $0 < (1+\xi)^{-3/2} < 1$. The required inequality follows by noting that

$$\begin{aligned} 1 + \frac{1}{2}x - \frac{1}{8}x^2 =& f(0) + f'(0)x + \frac{1}{2!} \cdot \frac{-1}{4}x^2 \\ \leq& f(0) + f'(0)x + \frac{1}{2!}f''(\xi)x^2 = f(x) = \sqrt{1+x} \\ \leq& f(0) + f'(0)x + \frac{1}{2!} \cdot 0x^2 \\ =& 1 + \frac{1}{2}x \end{aligned}$$

Comment. You should compare the assumption of the Taylor's Theorem to that of the Mean Value Theorem and observe the similarity.