MATH 2050A - Home Test 1 - Solutions

Suggested Solutions*(It does not reflect the marking scheme)*

1. (a) Use the ε-N definition to show $\lim \frac{2n^2-1}{n^2-n+1} = 2$.

   (b) Use the ε-N definition to show that the sequence $x_n = \frac{(-1)^n n}{n-1}, n = 2, 3...$ is divergent.

   (c) For each $x \in (0,1) \cap \mathbb{Q}$, put $f(x) = \frac{1}{b}$ if $x = \frac{a}{b}$, where a, b are positive integers and the H.C.F. of a and b is 1, i.e., a and b are relatively prime.
   Write $(0,1) \cap \mathbb{Q} = \{x_1, x_2, x_3, ...\}$.

   (i) Show that the sequence $(f(x_n))$ is convergent. Use the ε-N definition to justify your answer.

   (ii) Show that the limit of $(f(x_n))$ dose not depend on the order of the sequence: $x_1, x_2, x_3, ...$; i.e., if $\tau : \{1, 2...\} \to \{1, 2...\}$ is a bijection, then $\lim f(x_n) = \lim f(x_{\tau(n)})$. Explain your answer.

Solution.

(a) Let $\epsilon > 0$. By the Archimedean Property, take $N \in \mathbb{N}$ such that $N - 1 > \max\{1/\epsilon, 1\}$. When $n \geq N$, we have

$$\left|\frac{2n^2-1}{n^2-n+1} - 2\right| = \left|\frac{2n-3}{n^2-n+!1}\right| = \frac{2n-3}{n^2-n+1} \leq \frac{2n}{n^2-n} = \frac{2}{n-1} \leq \frac{2}{N-1} \leq 2\epsilon$$

where the second inequality has made use of the fact that $n \geq N \geq 2$. The result follows from the $\epsilon - N$ definition

(b) Note that for all $n \in \mathbb{N}$, we can rewrite $x_n = (-1)^n + \frac{(-1)^n}{n-1}$. We proceed to prove by contradiction.
Suppose (x_n) converges to some $L \in \mathbb{R}$. Then by the $\epsilon - N$ definition of convergence, there exists $N \in \mathbb{N}$ and $N \geq 2$ such that for all $n \geq N$, we have $|x_n - L| < 1/4$. Now we take $m \geq N$ an odd number. Then we have

$$|x_{m+1} - x_m| \leq |x_m - L| + |x_{m+1} - L| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Nonetheless, we have

$$|x_{m+1} - x_m| = \left|1 + \frac{1}{m+1-1} - (-1 + \frac{-1}{m-1})\right| = \left|2 + \frac{1}{m} + \frac{1}{m-1}\right| = 2 + \frac{1}{m} + \frac{1}{m-1} \geq 2$$

Contradiction arises as $2 < 1/2$.

(c.i) We claim that $\lim f(x_n) = 0$.
First, let $\eta : \mathbb{Q} \cap (0,1) \to \mathbb{N}$ be the bijection induced by the enumeration of rational numbers in the question, that is, $\eta(x_n) = n$ for all $n \in \mathbb{N}$. Next, we define $\mathcal{F}_n := \{q \in \mathbb{Q} : f(q) \geq 1/n\}$ for all $n \in \mathbb{N}$. We proceed to claim that $\mathcal{F}_n$ are all finite sets. Since for all $q \in \mathbb{Q} \cap (0,1)$ and $n \in \mathbb{N}$, we have $f(q) = 1/n$ if and only if $q = m/n$ for some $1 \leq m \leq n$ with $\gcd(m,n) = 1$. (The if part follows from the definition of f while the only if part follows from the fact that every fraction has a unique irreducible form, that is with coprime numerator and positive denominator). Hence for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathcal{F}_n &= \{q \in \mathbb{Q} : f(q) \geq 1/n\} \\ &= \{q \in \mathbb{Q} : f(q) = 1, \ldots, \frac{1}{n}\} = \bigcup_{i=1}^{n} \{q \in \mathbb{Q} : f(q) = \frac{1}{n}\} = \bigcup_{i=1}^{n} \{\frac{m}{n} \Big| \gcd(m,n) = 1, 1 \leq m \leq n\}\end{aligned}$$

So we have the cardinality $|\mathcal{F}_n| = \sum_{i=1}^{n} \phi(i)$ where ϕ is the Euler's Totient function with $\phi(i)$ denoting the number of natural numbers in $[1, i]$ that is coprime to i. Therefore $\mathcal{F}_n$ is a finite set for all $n \in \mathbb{N}$.
Now we are ready with the $\epsilon - N$ proof for the convergence. Let $\epsilon > 0$. By the Archimedean Property, we take $k \in \mathbb{N}$ such that $k > 1/\epsilon$. Since $\mathcal{F}_k$ is a finite set, it is possible to define $N := \max_{q \in \mathcal{F}_k} \eta(q)$. Now suppose $n \geq N + 1$, then $x_n \notin \mathcal{F}_k$. Otherwise, if we had $x_n \in \mathcal{F}_k$, then $N \geq \eta(x_n) = n \geq N + 1$, which is a contradiction. Hence by definition of $\mathcal{F}_k$, we have $|f(x_n) - 0| = f(x_n) < 1/k \leq \epsilon$. The result follows from the $\epsilon - N$ definition of sequential convergence.

(c.ii) We use the same $\eta, \mathcal{F}_n$ as defined in part (i). Let $\tau : \mathbb{N} \to \mathbb{N}$ be a bijection. We claim that $\lim f(x_{\tau(n)}) = 0$. Let $\epsilon > 0$. By the Archimedean Property, we take $k \in \mathbb{N}$ with $k > 1/\epsilon$. By finiteness of $\mathcal{F}_k$, we can take $N := \max_{q \in \mathcal{F}_k} \tau^{-1}\eta(q)$. Now suppose $n \geq N + 1$. Then $x_{\tau(n)} \notin \mathcal{F}_k$. Otherwise, if we had $x_{\tau(n)} \in \mathcal{F}_k$, then $N \geq \tau^{-1}\eta(x_{\tau(n)}) = \tau^{-1}\tau(n) = n \geq N + 1$, which is a contradiction. Hence by definition of $\mathcal{F}_k$, we have $\left|f(x_{\tau(n)}) - 0\right| = f(x_{\tau(n)}) < 1/k \leq \epsilon$. The result follows from the $\epsilon - N$ definition of sequential convergence.

2. Let $\rho : \mathbb{R} \to [0, \infty)$ be a non-zero function, i.e., $\rho(a) \neq 0$ for some $a \in \mathbb{R}$. Suppose that it satisfies the following conditions.

   (a) $\rho(0) = 0$.

   (b) $\rho(x - y) \leq \rho(x) + \rho(y)$ for all $x, y \in \mathbb{R}$.

   (c) $\rho(xy) \leq \rho(x)\rho(y)$ for all $x, y \in \mathbb{R}$.

   Show the following assertions.

   (i) We say that a number L is a ρ-limit of a sequence (x_n) if $\lim \rho(x_n - L) = 0$. Then such number L is unique if it exists. In this case, we call (x_n) a ρ-convergent sequence.

   (ii) We say that (x_n) is a ρ-Cauchy sequence if for any $\varepsilon > 0$, there is a positive integer N such that $\rho(x_m - x_n) < \varepsilon$ for all $m, n \geq N$.
   If (x_n) is a ρ-Cauchy sequence, then $\lim \rho(x_n)$ exists.

   (iii) Give an example of ρ to show that a Cauchy sequence (as defined in the text book) need not be a ρ-Cauchy sequence.
   Also, using this example to explain why a convergent sequence is not necessary to be a ρ-convergent sequence.

   (iv) i. If (x_n) and (y_n) both are ρ-Cauchy sequences, then so are the sequences $(x_n + y_n)$ and $(x_n \cdot y_n)$.
   ii. If we further assume that a function ρ satisfies $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in \mathbb{R}$, then (x_n^{-1}) is also a ρ-Cauchy sequence when (x_n) is a ρ-Cauchy sequence with $x_n \neq 0$ for all n and $\lim \rho(x_n) \neq 0$.

Solution. We first show that ρ satisfies three additional properties (d) - (f):

(d) *For all $x \in \mathbb{R}$, if $\rho(x) = 0$ then $x = 0$.* Suppose not. Then there exists $x \neq 0$ with $\rho(x) = 0$. Note x has an multiplicative inverse $x^{-1} \in \mathbb{R}$. Therefore we have $\rho(1) = \rho(xx^{-1}) \leq \rho(x)\rho(x^{-1}) = 0$ as $\rho(x) = 0$. As ρ is non-negative, we have $\rho(1) = 0$. Therefore for all $a \in \mathbb{R}$, we have $\rho(a) = \rho(a \cdot 1) \leq \rho(a)\rho(1) = 0$. Hence $\rho(a) = 0$ for all $a \in \mathbb{R}$ by non-negativity of ρ. However, ρ is non-zero. Contradiction arises.

(e) *For all $x \in \mathbb{R}$, we have $\rho(x) = \rho(-x)$.* Let $x \in \mathbb{R}$. By the second property of ρ, we have $\rho(0 - x) \leq \rho(0) + \rho(x)$. Hence $\rho(-x) \leq \rho(x)$. Since the previous inequality holds for arbitrary real numbers and $-x \in \mathbb{R}$, by replacing x with $-x$, we have $\rho(x) = \rho(-(-x)) \leq \rho(-x)$. By the partial ordering property we have $\rho(x) = \rho(-x)$.

(f) *For all $x, y \in \mathbb{R}$, the triangle inequality holds, that is, $\rho(x + y) \leq \rho(x) + \rho(y)$.* Let $x, y \in \mathbb{R}$. Then property (b) and (e) of ρ, we have $\rho(x + y) = \rho(x - (-y)) \leq \rho(x) + \rho(-y) = \rho(x) + \rho(y)$.

We then proceed to do the question.

(i). Let (x_n) be a sequence and L_1, L_2 be such that $\lim \rho(x_n - L_1) = \lim \rho(x_n - L_2) = 0$. Note that by triangle inequality, we have for all $n \in \mathbb{N}$, $\rho(L_1 - L_2) \leq \rho(L_1 - x_n) + \rho(x_n - L_2) = \rho(x_n - L_1) + \rho(x_n - L_2)$. By the sum law of limit, we have $\lim \rho(x_n - L_2) + \rho(x_n - L_1) = 0 + 0 = 0$. Since ρ is non-negative, by the sandwich theorem, $\lim \rho(L_1 - L_2) = 0$. Hence $\rho(L_1 - L_2) = 0$. By the first additional property, we have $L_1 - L_2 = 0$. Hence $L_1 = L_2$ and hence ρ limits are unique.

(ii). Let (x_n) be ρ Cauchy. By the Cauchy Criteria, it suffices to show that $(\rho(x_n))$ is a Cauchy sequence.
Let $\epsilon > 0$ then there exists $N \in \mathbb{N}$ such that $n, m \geq N$ imples $\rho(x_n - x_m) < \epsilon$. Now suppose $n, m \geq N$, then by the triangle inequality of ρ (property (f)), we have $\rho(x_n) = \rho(x_n - x_m + x_m) \leq \rho(x_n - x_m) + \rho(x_m)$ and $\rho(x_m) \leq \rho(x_m - x_n) + \rho(x_n)$. Combining the two inequalities, we have

$$|\rho(x_n) - \rho(x_m)| \leq \rho(x_n - x_m) < \epsilon$$

By definition of Cauchy sequences, $(\rho(x_n))$ is a Cauchy sequence.

(iii). We consider the trivial $\rho : \mathbb{R} \to [0, \infty)$, that is, $\rho(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$ for all $x \in \mathbb{R}$. Note that ρ is a non-zero function. We proceed to show that the trivial ρ satisfies the three properties in the question.

(a) $\rho(0) = 0$ is clear.

(b) Let $x, y \in \mathbb{R}$. There is nothing to prove if $\rho(x - y) = 0$ as ρ is non-negative. Suppose $\rho(x - y) = 1$. Then by definition $x - y \neq 0 \Rightarrow x \neq y$. Hence at least one of x, y is non-zero. Suppose x is non-zero without loss of generality. Then $\rho(x - y) = 1 = \rho(x) \leq \rho(x) + \rho(y)$.

(c) Let $x, y \in \mathbb{R}$. There is nothing to prove if $\rho(xy) = 0$. Suppose $\rho(xy) = 1$. Then $xy \neq 0$. Hence $x, y \neq 0$ (since $\mathbb{R}$ is a field which is an integral domain). So, $\rho(x) = \rho(y) = 1$. Then $\rho(xy) = 1 \leq \rho(x)\rho(y)$.

Now consider the sequence $(1/n)$. Then $(1/n)$ converges and hence is Cauchy in the ordinary sense. However $(1/n)$ is not ρ Cauchy where ρ is the trivial one. It is because for all $N \in N$, we can take $n := N, m := N + 1$, then $n \neq m$. Hence $\rho(x_n - x_m) = 1$. So $(1/n)$ is not ρ Cauchy by the negation of its definition.
Next, as $(1/n)$ is a convergent sequence, it remains to show that $(1/n)$ is not ρ convergent. This follows once we have showed that every ρ convergence sequence is ρ Cauchy:
Let (x_n) be a ρ convergent sequence. Then there exists $L \in \mathbb{R}$ such that $\lim \rho(x_n - L) = 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\rho(x_n - L) < \epsilon$ for all $n \geq N$. Hence for all $n, m \geq N$, we have by triangle inequality that $\rho(x_n - x_m) \leq \rho(x_n - L) + \rho(L - x_m) < 2\epsilon$. It follows that (x_n) is ρ Cauchy. Hence $(1/n)$ is not ρ convergent as it is not ρ Cauchy.

(iv). (a) Let $(x_n), (y_n)$ be ρ Cauchy sequences. We claim that their sum and product are also ρ Cauchy.
Sum: Let $\epsilon > 0$. Then there exists $N_x, N_y \in \mathbb{N}$ such that

$$\rho(x_n - x_m) < \epsilon \text{ when } n, m \geq N_x$$
$$\rho(y_n - y_m) < \epsilon \text{ when } n, m \geq N_y$$

We pick $N := \max\{N_x, N_y\}$. Then for all $n, m \geq N$, we have by triangle inequality,

$$\rho(x_n + y_n - x_m - y_m) \leq \rho(x_n - x_m) + \rho(y_n - y_m) < 2\epsilon$$

The result follows by definition.
Product: First by part (ii), as $(x_n), (y_n)$ are ρ Cauchy, $((\rho(x_n)), (\rho(y_n))$ are convergent and hence bounded. Let $M_x, M_y \in \mathbb{R}$ be such that $\rho(x_n) \leq M_x$ and $\rho(y_n) \leq M_y$ for all $n \in \mathbb{R}$. Now let $\epsilon > 0$, we define N_x, N_y, N the same way as we do in the proof concerning sums. Then for all $n, m \geq N$, we have by the triangle inequality,

$$\begin{aligned}\rho(x_n y_n - x_m y_m) &= \rho(x_n y_n - x_n y_m + x_n y_m - x_m y_m) \\ &\leq \rho(x_n)\rho(y_n - y_m) + \rho(y_m)\rho(x_n - x_m) \leq M_x \rho(y_n - y_m) + M_y \rho(x_n - x_m) < (M_x + M_y)\epsilon\end{aligned}$$

It follows that $(x_n y_n)$ is ρ Cauchy.

(b) Now ρ satisfies that $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in \mathbb{R}$. We first show the following additional properties of ρ.

(g) $\rho(1) = 1$. Since $\rho(1) = \rho(1 \cdot 1) = \rho(1)\rho(1)$, we have $\rho(1) = 0$ or $\rho(1) = 1$. By the first additional property in the beginning, the former is impossible as $1 \neq 0$. Hence $\rho(1) = 1$

(h) $\rho(x^{-1}) = (\rho(x))^{-1}$ for all $x \neq 0$. Let $x \neq 0$. Then $1 = \rho(1) = \rho(xx^{-1}) = \rho(x)\rho(x^{-1)}$. The result follows clearly.

Now we are ready to do the question. Let (x_n) be a ρ Cauchy sequence with $x_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim \rho(x_n) \neq 0$. We proceed to show (x_n^{-1}) is a ρ Cauchy sequence.
Let $\epsilon > 0$. By non-negativity of ρ and order preserving property of limit, we have $\lim \rho(x_n) \geq 0$. Denote $L := \lim \rho(x_n) > 0$ (since $\lim \rho(x_n) \neq 0$). Then $\sup_n \inf_{k \geq n} \rho(x_k) = \underline{\lim}\, \rho(x_n) = \lim \rho(x_n) > L/2$. By definition of supremum, there exists $K_L \in \mathbb{N}$ such that $\inf_{k \geq K_L} \rho(x_k) > L/2$, and so $\rho(x_n) > L/2$ if $n \geq K_L$. Since (x_n) is ρ Cauchy, there exists $N_0 \in \mathbb{N}$ such that $\rho(x_n - x_m) < \epsilon$ for $n, m \geq N_0$.
Now take $N := \max\{K_L, N_0\}$. Then for all $n, m \geq N$, we have by the assumption and the new property (h),

$$\rho(\frac{1}{x_n} - \frac{1}{x_m}) = \rho(\frac{x_m - x_n}{x_n x_m}) = \rho(x_m - x_n)\rho((x_n x_m)^{-1}) = \frac{\rho(x_m - x_n)}{\rho(x_n x_m)} = \frac{\rho(x_n - x_,)}{\rho(x_n)\rho(x_m)} \leq \frac{4}{L^2}\epsilon$$

The result follows by definition of ρ Cauchy sequence.