MATH 2050A - Home Test 2 - Solutions

Suggested Solutions *(It does not reflect the marking scheme)*

1. (a) Use the ε-δ notation to show that the function $f(x) = \frac{x^3+1}{x^2+1}$ is continuous on $[0,1]$.
   (b) Let m, n be positive integers. Find
$$\lim_{x\to 0} \frac{(1+mx)^n - (1+nx)^m}{x^2}.$$
   Use the ε-δ notion to justify your answer.
   (c) Let $[x]$ be the integral part of x, i.e. $[x] := \max\{m \in \mathbb{Z} : m \le x\}$.
   Use the ε-δ notion to determine whether $\lim_{x\to 0} x[x^{-1}]$ exists or not.

Solution.

(a) Notice that for any $x, u \in [0,1]$, we have
$$f(x) - f(u) = \frac{x^3+1}{x^2+1} - \frac{u^3+1}{u^2+1} = \frac{(x-u)(x^2u^2 + x^2 + xu + u^2 - x - u)}{(x^2+1)(u^2+1)}.$$
Also, whenever $x, u \in [0,1]$, we have $x^2+1 \ge 1$, $u^2+1 \ge 1$, and
$$|x^2u^2 + x^2 + xu + u^2 - x - u| \le |x|^2|u|^2 + |x||u| + |x|^2 + |u|^2 + |x| + |u| \le 6.$$
In this case,
$$|f(x) - f(u)| = \frac{|x^2u^2 + x^2 + xu + u^2 - x - u|}{(x^2+1)(u^2+1)} \cdot |x-u| \le 6|x-u|.$$
Let $x \in [0,1]$ and $\varepsilon > 0$. Take $\delta = \varepsilon/6$. Then whenever $u \in [0,1]$ and $|x-u| < \delta$,
$$|f(x) - f(u)| \le 6|x-u| < 6\delta = \varepsilon.$$
It follows by definition that f is continuous on $[0,1]$.

(b) The limit is given by
$$\lim_{x\to 0} \frac{(1+mx)^n - (1+nx)^m}{x^2} = \frac{1}{2}mn(n-m).$$
To see this, let $k = \max\{m, n, 3\}$. Notice that by the **Binomial Theorem**, we have
$$(1+mx)^n = \sum_{i=0}^{k} a_i x^i \quad \text{and} \quad (1+nx)^m = \sum_{i=0}^{k} b_i x^i,$$
where a_i and b_i are constants given by
$$a_i = \binom{n}{i} m^i \quad \text{and} \quad b_i = \binom{m}{i} n^i.$$
Here, we adopt the convention
$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, \quad \text{if } 0 \le r \le n \quad \text{and} \quad \binom{n}{r} = 0 \quad \text{otherwise.}$$
By direct computations, we always have $a_0 = b_0$ and $a_1 = b_1$. Hence
$$\frac{(1+mx)^n - (1+nx)^m}{x^2} = c_2 + c_3 x + \cdots + c_k x^{k-2}, \quad \forall x \in \mathbb{R} \setminus \{0\}.$$
Here, $c_i = a_i - b_i$ for $i = 0, 1, ..., k$. Moreover, notice that if $|x| \le 1$, we have $|x|^i \le |x|$ for all $i \in \mathbb{N}$. In this case, we can estimate
$$\left| \frac{(1+mx)^n - (1+nx)^m}{x^2} - c_2 \right| \le |c_3||x| + \cdots + |c_k||x|^{k-2} \le C|x|, \tag{1}$$
where $C = |c_3| + \cdots + |c_k| \ge 0$. With these preparations, we proceed to prove the result. Consider the following two cases:

- $C = 0$. Let $\varepsilon > 0$ and take $\delta = 1$. Then by (1), we have

$$\left| \frac{(1+mx)^n - (1+nx)^m}{x^2} - c_2 \right| = 0 < \varepsilon, \quad \text{whenever } 0 < |x| < \delta.$$

- $C > 0$. Let $\varepsilon > 0$ and take $\delta = \min\{1, \varepsilon/C\}$. Then by (1), we have

$$\left| \frac{(1+mx)^n - (1+nx)^m}{x^2} - c_2 \right| \leq C|x| < C\delta \leq \varepsilon, \quad \text{whenever } 0 < |x| < \delta.$$

In any cases, it follows by definition that the limit is given by

$$c_2 = a_2 - b_2 = \frac{1}{2}mn(n-m).$$

Remark. The case $C = 0$ occurs when $m \leq 2$ and $n \leq 2$ since the coefficients a_i and b_i are all zero for $i = 3, ..., k$. We also need to consider these special cases when we calculate c_2. It just appears that they enjoy the same formula with the general case.

(c) The limit exists and is given by

$$\lim_{x \to 0} x[x^{-1}] = 1.$$

Consider the *fractional part* of x, defined by $\{x\} = x - [x]$. We first claim that

$$0 \leq \{x\} < 1, \quad \forall x \in \mathbb{R}.$$

To see this, we have $[x] \leq x$ by the definition of $[x]$. Hence $\{x\} = x - [x] \geq 0$. On the other hand, suppose on a contrary that $\{x\} = x - [x] \geq 1$. This implies $x \geq 1 + [x]$ and hence $[x] \geq 1 + [x]$ by the definition of $[x]$, which is absurd. The claim follows.
We proceed to prove the result. Notice that

$$x[x^{-1}] - 1 = x([x^{-1}] - x^{-1}) = -x \cdot \{x^{-1}\}, \quad \forall x \in \mathbb{R} \setminus \{0\}.$$

Let $\varepsilon > 0$ and take $\delta = \varepsilon$. Then whenever $0 < |x| < \delta$,

$$\left| x[x^{-1}] - 1 \right| = \left| x\{x^{-1}\} \right| = |x| \cdot \left| \{x^{-1}\} \right| < |x| \cdot 1 < \delta = \varepsilon.$$

It follows by definition that the limit is given by 1.

2. Let $P(x)$ a real polynomials function of degree n, denoted its degree by $deg(P) = n$, defined on $\mathbb{R}$, i.e., $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_k \in \mathbb{R}$ and $a_n \neq 0$.

   (a) By using the definition of limits of functions, show that $\lim_{x\to\infty} |P(x)| = \infty$ if P is a non-constant polynomial.

   (b) Is $P(x)$ uniformly continuous on $\mathbb{R}$?

   (c) Let $Q(x)$ be a polynomial function. Suppose that $Q(x) \neq 0$ for all $x \in \mathbb{R}$. Let $R(x) = \dfrac{P(x)}{Q(x)}$. Describe whether the rational function $R(x)$ is uniformly continuous on $\mathbb{R}$ in the following three cases: (i) : $deg(P) > deg(Q)$; (ii) : $deg(P) = deg(Q)$ and (iii) : $deg(P) < deg(Q)$. Explain you answer.

   Solution.

(a) Let $P(x)$ be a non-constant polynomial. We write $P(x) = \sum_{i=0}^n a_i x^i$ where $n \geq 1$ and $a_n \neq 0$. We first show that

$$\lim_{x\to\infty} \frac{\sum_{i=0}^{n-1} a_i x^i}{a_n x^n} = 0$$

This is because we have for all $x \neq 0$

$$\frac{\sum_{i=0}^{n-1} a_i x^i}{a_n x^n} = \sum_{i=0}^{n-1} \frac{a_i}{a_n} \frac{1}{x^{n-i}} = \frac{a_0}{a_n}\frac{1}{x^n} + \cdots + \frac{a_{n-1}}{a_n}\frac{1}{x}$$

(Note that the fact $n \geq 1$ has been used to ensure there are x terms with negative orders.) The results follow by applying the sum, product and scalar law of limits on $\lim_{x\to\infty} \frac{1}{x} = 0$.
By the limits proved above, there exists $A > 0$ such that

$$\left| \frac{\sum_{i=0}^{n-1} a_i x^i}{a_n x^n} \right| \leq \frac{1}{2} \iff \left| \sum_{i=0}^{n-1} a_i x^i \right| \leq \frac{|a_n x^n|}{2}$$

if $x > A$.
Now let $M > 0$. We then take $R := \max\{(\frac{2M}{|a_n|})^{1/n}, A\} > 0$. Then we have by the triangle inequality and the above approximations that

$$\begin{aligned} |P(x)| &= \left| \sum_{i=0}^{n} a_i x^i \right| \\ &\geq |a_n x^n| - \left| \sum_{i=0}^{n-1} a_i x_i \right| \geq |a_n x^n| - \frac{|a_n x^n|}{2} = \frac{1}{2}|a_n||x|^n \geq \frac{1}{2}|a_n|\frac{2M}{|a_n|} = M \end{aligned}$$

when $x > R$ Hence by definition of limits, we have $\lim_{x\to\infty} |P(x)| = \infty$.

(b) We claim that $P(x)$ is uniformly continuous on $\mathbb{R}$ if and only if $n := \deg(P(x)) = 0$ or 1.
$(n = 0{:})$ In this case $P(x)$ is a constant polynomial. The result is clear.
$(n = 1{:})$ We write $P(x) = a_1 x + a_0$ where $a_1 \neq 0$. Then for all $x, y \in \mathbb{R}$, we have

$$|P(x) - P(y)| = |a_1(x - y)| = |a_1||x - y|$$

By definition, $x \mapsto P(x)$ is a Lipschitz function on $\mathbb{R}$. Therefore $P(x)$ is uniformly continuous on $\mathbb{R}$

$(n \geq 2:)$. We write $P(x) = \sum_{i=0}^{n} a_i x^i$ where $a_n \neq 0$. We proceed to show that $x \mapsto P(x)$ is not uniformly continuous by definition. Take $\epsilon := 1$. Let $\delta > 0$ be arbitrary. We define a polynomial by $Q_\delta(x) := P(x+\delta/2) - P(x)$. By using the algebraic identity $a^k - b^k = (a-b)\sum_{i=0}^{k-1} a^{k-1-i}b^i$ where $a, b \in \mathbb{R}$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} Q_\delta(x) &= P(x+\delta/2) - P(x) \\ &= \sum_{k=1}^{n} a_k \left[(x+\frac{\delta}{2})^k - x^k \right] \\ &= \sum_{k=1}^{n} a_k \frac{\delta}{2} \left[(x+\frac{\delta}{2})^{k-1} + (x+\frac{\delta}{2})^{k-2}x^1 + \cdots + x^{k-1} \right] \end{aligned}$$

We could see that Q_δ has a non-constant term $a_n \frac{\delta}{2} n x^{n-1}$ (as $n \geq 2$). Hence Q_δ is a non-constant polynomial. By the result of part (a), we have $\lim_{x\to\infty} |Q_\delta(x)| = \infty$. By the definition of the limit, there exists $R > 0$ such that when $x > R$ we have $|Q_\delta(x)| \geq \epsilon := 1$.
Now we take $x_\delta := R+1$ and $y_\delta := R+1+\frac{\delta}{2}$. Then $|x_\delta - y_\delta| < \delta$. However,

$$|P(x_\delta) - P(y_\delta)| = \left| P(R+1) - P(R+1+\frac{\delta}{2}) \right| = |Q_\delta(R+1)| \geq 1$$

The result follows by the definition of uniform continuity.
Remark. The construction of Q_δ for $n \geq 2$ is motivated by considering derivatives of polynomials. (Although we probably won't allow you to use differentiation techniques, you can always draw intuitions from them. Such thoughts help in later questions as well).

(c) We proceed to do the question in *reverse order* and we would be using the following facts:

Proposition 0.1 (Tutorial 10 - Example 4). *(Can be used without proof).*
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$$

Then f is uniformly continuous on $\mathbb{R}$.

Proposition 0.2 (Division Algorithm for Polynomials). *(Can be used without proof).*
Let $f(x), g(x)$ be real polynomials where $g(x) \neq 0$. Then there exist real polynomials $q(x)$ and $r(x)$ (called quotient and remainder respectively) where $\deg R(x) < \deg g(x)$ or $r(x) = 0$ such that $f(x) = g(x)q(x) + r(x)$.

Remark. The proofs of these Propositions are left as Exercise. In fact in Proposition 0.1, the assumption could be weakened to the existence instead of the equality of the limits. Note that the field axioms of $\mathbb{R}$ plays a crucial role in the proof of Division Algorithm.

(iii). $(\deg(P) < \deg(Q))$. Yes they are uniformly continuous. Since polynomials are continuous and $Q(x) \neq 0$ for all $x \in \mathbb{R}$. By the quotient law for continuity, $R(x) := P(x)/Q(x)$ is continuous on $\mathbb{R}$. It remains to show that $\lim_{x\to\infty} R(x) = \lim_{x\to-\infty} R(x) = 0$ and apply Proposition 0.1 above.
Now we write $P(x) = \sum_{i=0}^{n} a_i x^i$ and $Q(x) = \sum_{i=0}^{m} b_i x^i$ where $m > n \geq 0$, $b_i \in \mathbb{R}$ for $i = 0, \cdots, m$ with $b_m \neq 0$ and $a_j \in \mathbb{R}$ for $j = 0, \cdots, n$ with $a_n \neq 0$. We observe that for $x \neq 0$, we have

$$\begin{aligned} \frac{P(x)}{Q(x)} &= \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i} = \frac{x^{-m}\sum_{i=0}^{n} a_i x^i}{x^{-m}\sum_{i=0}^{m} b_i x^i} = \frac{\sum_{i=0}^{n} a_i x^{i-m}}{\sum_{i=0}^{m} b_i x^{i-m}} \\ &= \frac{\dfrac{a_0}{x^m} + \cdots + \dfrac{a_n}{x^{m-n}}}{\dfrac{b_0}{x^m} + \cdots + \dfrac{b_{m-1}}{x} + b_m} \end{aligned}$$

By applying the algebraic limits laws on $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$, it follows that $\lim_{x\to\infty} \frac{P(x)}{Q(x)} = 0 = \lim_{x\to-\infty} \frac{P(x)}{Q(x)}$.
Therefore, the condition for the Proposition 0.1 is fulfilled and we conclude that $R(x)$ is uniformly continuous on $\mathbb{R}$.

(ii). $(\deg(P) = \deg(Q))$. Yes, they are uniformly continuous. By the division algorithm of real polynomials, we write $P(x) = f(x)Q(x) + r(x)$ where $f(x), r(x)$ are real polynomials where $\deg r(x) < \deg Q(x)$. Furthermore $\deg(f) = 0$ (that is f is a constant polynomial) by comparing degrees of the expression. Therefore we have

$$R(x) = \frac{P(x)}{Q(x)} = f(x) + \frac{r(x)}{Q(x)}$$

By results of previous parts, $f(x)$ and $\frac{r(x)}{R(x)}$ are uniformly continuous on $\mathbb{R}$. Therefore $R(x)$ is uniformly continuous on $\mathbb{R}$ as the sum of uniformly continuous functions is again uniformly continuous.

(i). $(\deg(P) > \deg(Q))$ We claim that $R(x)$ is uniformly continuous on $\mathbb{R}$ if and only if $\deg(P) = \deg(Q) + 1$.
(If readers understand the argument in part (ii), they should see that this fact follows immediately from similar arguments. Below is the full argument).
$(\deg(P) = \deg(Q) + 1)$: By the division algorithm of real polynomials, we write $P(x) = f(x)Q(x) + r(x)$ where $f(x), r(x)$ are real polynomials where $\deg r(x) < Q(x)$. Furthermore $\deg(f) = 1$ (that is f is a linear polynomial) by comparing degrees of the expression. Since linear polynomials are uniformly continuous by part (b), the result follows as $R(x)$ is sum of uniformly continuous functions.
$(\deg(P) \geq \deg(Q) + 2)$: By the division algorithm of real polynomials, we write $P(x) = f(x)Q(x) + r(x)$ where $f(x), r(x)$ are real polynomials where $\deg r(x) < \deg Q(x)$. Furthermore $\deg(f) \geq 2$ by comparing degrees of the expression. Therefore we have $R(x) = \frac{P(x)}{Q(x)} = f(x) + \frac{r(x)}{Q(x)}$.
Suppose $R(x)$ were uniformly continuous on $\mathbb{R}$. Then $f(x)$ is uniformly continuous on $\mathbb{R}$ as we can write that $f(x) = \frac{r(x)}{Q(x)} - R(x)$. However, $f(x)$ has degree ≥ 2. By part (b), contradiction arises. It follows then $R(x)$ is not uniformly continuous on $\mathbb{R}$.

3. Let f be a real-valued function defined on a non-empty subset A of $\mathbb{R}$. f is said to be locally bounded at a point $z \in \mathbb{R}$ (x is not necessary in A) if there is $r > 0$ such that f is bounded on $(z - r, z + r) \cap A$. Furthermore, if z is a limit point of A, then we put

$$L(z) := \inf_{r>0} \sup\{f(x) : 0 < |x - z| < r\} \text{ and } l(z) := \sup_{r>0} \inf\{f(x) : 0 < |x - z| < r\}.$$

(a) Assume that f is locally bounded at a limit point z of A. Prove or disprove the following statement: "f has the limit at z if and only if $L(z) = l(z)$".

(b) Show that if A is compact and f is locally bounded at every point in A, then f is bounded. Give a counter-example of f to show that the assumption of compactness of A cannot be removed.

(c) Give an example of a continuous function f defined on $(0, 1)$ which is not locally bounded at 0 but $\lim_{x\to 0} |f(x)| \neq \infty$.

Solution.

(a) The statement is correct. To simplify notations, define for each $r > 0$,

$$E_r = \Big\{f(x) : 0 < |x - z| < r \text{ and } x \in A\Big\}.$$

With this notation, we have

$$\ell(z) = \sup_{r>0} \inf E_r \quad \text{and} \quad L(z) = \inf_{r>0} \sup E_r.$$

We first claim that $\ell(z) \leq L(z)$ in any cases. Let $\varepsilon > 0$. Then by definition of $\ell(z)$ and $L(z)$, there exist $r, s > 0$ such that

$$\inf E_r \geq \ell(z) - \frac{\varepsilon}{2} \quad \text{and} \quad \sup E_s \geq L(z) + \frac{\varepsilon}{2}. \tag{2}$$

- If $r \leq s$, we have $E_r \subseteq E_s$, which implies that $\inf E_r \leq \sup E_r \leq \sup E_s$.
- If $s \leq r$, we have $E_s \subseteq E_r$, which implies that $\inf E_r \leq \inf E_s \leq \sup E_s$.

In any cases, $\ell(z) \leq L(z) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the claim follows. ($\Rightarrow$) Suppose $\lim_{x\to z} f(x) = y$. Let $\varepsilon > 0$. By definition of limits, there exists $r > 0$ such that whenever $x \in A$ and $|x - z| < r$, $|f(x) - y| < \varepsilon$. i.e., $E_r \subseteq (y - \varepsilon, y + \varepsilon)$. Hence

$$y - \varepsilon \leq \inf E_r \leq \ell(z) \leq L(z) \leq \sup E_r \leq y + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\ell(z) = L(z) = y$. ($\Leftarrow$) Suppose $\ell(z) = L(z) = y$. Let $\varepsilon > 0$. By definitions of $\ell(z)$ and $L(z)$, there exist $r, s > 0$ that satisfy (2). Take $\delta = \min\{r, s\}$. Then $E_\delta \subseteq E_r$ and $E_\delta \subseteq E_s$. Then whenever $x \in A$ and $|x - z| < \delta$,

$$y - \varepsilon \leq \inf E_r \leq \inf E_\delta \leq f(x) \leq \sup E_\delta \leq \sup E_s \leq y + \varepsilon.$$

It follows by definition that $\lim_{x\to z} f(x) = y$.

Remark. The locally boundedness of f at z guarantees the existence of $\ell(z)$ and $L(z)$.

(b) Since f is locally bounded at each point in A, for each $a \in A$, there exists $r_a > 0$ such that f is bounded on $(a - r_a, a + r_a) \cap A$. i.e., there exists $M_a > 0$ such that

$$|f(x)| \leq M_a, \quad \forall x \in (a - r_a, a + r_a) \cap A.$$

Notice that A is compact and $\{(a - r_a, a + r_a)\}_{a\in A}$ is an open intervals cover of A. The **Heine-Borel Property** yields $a_1, a_2, ..., a_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} (a_i - r_{a_i}, a_i + r_{a_i}).$$

We proceed to claim that f is bounded on A by $M = \max\{M_{a_1}, M_{a_2}, ..., M_{a_n}\}$. For each $x \in A$, there exists i such that $x \in (a_i - r_{a_i}, a_i + r_{a_i})$. Therefore $x \in (a_i - r_{a_i}, a_i + r_{a_i}) \cap A$ and hence $|f(x)| \leq M_{a_i} \leq M$.
For a counter-example to show that the assumption of compactness of A cannot be removed, consider $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = 1/x$. Notice that $(0, 1)$ is not compact and f is unbounded on $(0, 1)$. However, f is locally bounded at every point in $(0, 1)$. To see that f is locally bounded at $a \in (0, 1)$, take $r = a/2 > 0$. Then whenever $x \in (a - r, a + r) \cap (0, 1)$, we have $0 < a/2 < x < 3a/2$ and hence

$$\frac{2}{3a} < f(x) = \frac{1}{x} < \frac{2}{a}.$$

(c) Consider the continuous function $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x} \sin(1/x).$$

We first claim that this function is not locally bounded at 0. It suffices to find a sequence (x_n) in $(0, 1)$ such that $x_n \to 0$ and $f(x_n)$ is unbounded. This can be done by defining $x_n = (\pi/2 + 2n\pi)^{-1}$. Notice that $x_n \to 0$ as $n \to \infty$ and

$$f(x_n) = \frac{1}{(\pi/2 + 2n\pi)^{-1}} \sin(\pi/2 + 2n\pi) = \frac{\pi}{2} + 2n\pi.$$

We then define $y_n = (2n\pi)^{-1}$ for each $n \in \mathbb{N}$. Notice that $y_n \to 0$ as $n \to \infty$ and

$$|f(y_n)| = \frac{1}{(2n\pi)^{-1}} |\sin(2n\pi)| = 0, \quad \forall n \in \mathbb{N}.$$

Hence $\lim_{x\to 0} |f(x)|$ either equals zero or does not exists. In both cases, $\lim_{x\to 0} |f(x)| \neq \infty$.