MATH 2050A - HW 8 - Solutions

Commonly missed steps/key points in Purple

Solutions

1 (P.148 Q2)**.** Let $f(x) := 1/x^2$. Show that

i. f is uniformly continuous on $A := [1, \infty)$

ii. f is not uniformly continuous on $B := (0, \infty)$

Solution. i. Let $x, y \in A$. Then $x, y \geq 1$. Then we have

$$|f(x) - f(y)| = \frac{|x^2 - y^2|}{x^2y^2} = |x - y|\left|\frac{x + y}{x^2y^2}\right| \leq |x - y|\left|\frac{x + y}{xy}\right| = |x - y|\left|\frac{1}{y} + \frac{1}{x}\right| \leq 2|x - y|$$

in which we have used the fact that $x^2 \geq x$ if $x \geq 1$. Hence, f is Lipschitz continuous on A. It follows that f is uniformly continuous on A.

ii. Define $x_n := 1/n$ for $n \in \mathbb{N}$. Then (x_n) is a sequence in B which is Cauchy as it is convergent (even though the limit does not lie in B). However $f(x_n) = n^2$ for all $n \in \mathbb{N}$. So, $(f(x_n))$ is an unbounded sequence and so is not a Cauchy sequence. Because a uniformly continuous function should map Cauchy sequences into Cauchy seqeunces, we conclude that f is not uniformly continuous on B.

2 (P.148 Q6)**.** Let $A \subset \mathbb{R}$ and f, g be real-valued uniformly continuous functions defined on A. Show that if f, g are bounded on A, then the product fg is uniformly continuous on A

Solution. Let $M, N > 0$ be an upper bounded for f, g on A respectively. Now let $\epsilon > 0$, then there exists $\delta_1, \delta_2 > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ if } |x - y| < \delta_1$$
$$|g(x) - g(y)| < \epsilon \text{ if } |x - y| < \delta_2$$

Take $\delta := \min\{\delta_1, \delta_2\}$. Suppose $|x - y| < \delta$ with $x, y \in A$. Then we have

$$|f(x)g(x) - f(y)g(y)| \leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \leq (M + N)\epsilon$$

By definition of uniform continuity, fg is uniformly continuous on A

3 (P.148 Q7)**.** Let $f(x) := x$ and $g(x) := \sin x$ be defined on $\mathbb{R}$. Show that

i. f, g are uniformly continuous on $\mathbb{R}$

ii. the product fg is not uniformly continuous on $\mathbb{R}$

Solution. i. f: Let $x, y \in \mathbb{R}$. Then $|f(x) - f(y)| = |x - y| \leq |x - y|$. Therefore, f is Lipschitz continuous on $\mathbb{R}$. In particular, it is also uniformly continuous on $\mathbb{R}$.
g: Let $x, y \in \mathbb{R}$. Then by the sum-to-product formula (learnt in high school) we have

$$|g(x) - g(y)| = |\sin x - \sin y| = 2\left|\sin\left(\frac{x - y}{2}\right)\cos\left(\frac{x + y}{2}\right)\right| \leq 2\left|\sin\left(\frac{x - y}{2}\right)\right|$$

Suppose $|x - y| \geq 1$. Then $|g(x) - g(y)| \leq 2\left|\sin\left(\frac{x-y}{2}\right)\right| \leq 2 \leq 2|x - y|$.
Suppose $|x - y| < 1$ and WLOG take $x > y$ (since $\sin(-x) = -\sin x$). Then $0 \leq \frac{x-y}{2} \leq \pi/2$. By the fact that $\sin x \leq x$ for $x \geq 0$. We have

$$\left|\sin\left(\frac{x - y}{2}\right)\right| = \sin\left(\frac{x - y}{2}\right) \leq \frac{x - y}{2}$$

Therefore, $|g(x) - g(y)| \leq |x - y|$ if $|x - y| \leq 1$. Combining both cases ($|x - y| \geq 1$ and $|x - y| \leq 1$), we have that g is Lipschitz on $\mathbb{R}$. Hence g is uniformly continuous on $\mathbb{R}$.

ii. **Method 1: By definition of uniform continuity**
Suppose fg were uniformly continuous on $\mathbb{R}$. Then there exists $\delta > 0$ such that $|x - y| < \delta$ would imply $|f(x) - f(y)| < 1$. Now define $0 < \delta_0 := \min\{\delta, 1\}$

For all $n \in \mathbb{N}$, we define $x_n := 2n\pi + \frac{\delta_0}{4}$ and $y_n := 2n\pi - \frac{\delta_0}{4}$. Then $|x_n - y_n| = \delta_0/2 < \delta$ by construction, but we have

$$\begin{aligned} fg(x_n) - fg(y_n) &= (2n\pi + \delta_0)\sin(2n\pi + \delta_0) - (2n\pi - \delta_0)\sin(2n\pi - \delta_0) \\ &= (2n\pi + \delta_0)\sin(\delta_0) + (2n\pi - \delta_0)\sin(\delta_0) \\ &= 4n\pi\sin(\delta_0) \end{aligned}$$

Hence, $0 < |fg(x_n) - fg(y_n)| = 4\pi|\sin(\delta_0)|n < 1$ for all $n \in \mathbb{N}$ (note that $\sin(\delta_0) \neq 0$ by the choice of δ_0). By considering $n \to \infty$, it is clear that contradiction arises. Hence fg is not uniformly continuous on $\mathbb{R}$.

Method 2: Using a necessary condition for uniform continuity
We can use the following necessary condition for uniform continuity:

Proposition 0.1. *Let $f : A \to \mathbb{R}$ be uniformly continuous on some non-empty subset $A \subset \mathbb{R}$. Suppose (x_n) and (y_n) are two sequences in A such that $\lim_n x_n - y_n = 0$. Then we have $\lim_n f(x_n) - f(y_n) = 0$.*

The proof of the above is left as an exercise. Back to the question, we define $x_n := 2n\pi + \frac{1}{n}$ and $y_n := 2n\pi$ for all $n \in \mathbb{N}$. Then $\lim_n x_n - y_n = \lim_n \frac{1}{n} = 0$. However, we have

$$\begin{aligned} \lim_n fg(x_n) - fg(y_n) &= \lim_n (2n\pi + \frac{1}{n})\sin\left(2n\pi + \frac{1}{n}\right) - 2n\pi\sin(2n\pi) \\ &= \lim_n 2n\pi\sin\left(\frac{1}{n}\right) + \frac{1}{n}\sin\left(\frac{1}{n}\right) \\ &= 2\pi + 0 = 2\pi \end{aligned}$$

in which the first limit of the second row is computed by using sequential criteria on the limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$ (learnt in high school), together with the scalar compatibility of limits, and the second limit follows from the Squeeze Theorem.
Hence, by the contrapositive of the above necessary condition, we conclude that fg is not uniformly continuous.

Remark. You are reminded that we have not defined the sine function rigorously so until then you could use whatever property you learnt in high school about the sine function. Nowadays, trigonometric dunctions are usually defined in terms of power series, combinations of exponential functions or differential equations (as in the textbook) and so on. It is quite interesting that these new definitions retain properties of trigonometric functions as if they were defined using ratio of sides in a triangle.

Remark. For Q3ii, the sequences in (both) solutions is thought of by considering the graph of $fg(x) = x\sin x$. You could see that at those points in the solutions, the function is going faster as $n \to \infty$.