MATH 2050A - HW 7 - Solutions

Commonly missed steps in Purple

Solutions

1 (P.134 Q7)**.** Give an example of a function $f : [0,1] \to \mathbb{R}$ such that f is discontinuous at every point of $[0,1]$, but $|f|$ is continuous on $[0,1]$.

Solution. We define $f : [0,1] \to \mathbb{R}$ by $f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ -1 & x \in [0,1] \backslash \mathbb{Q} \end{cases}$. As $|f| = 1$, which is a constant function. It is clear that $|f|$ is continuous. It remains to show that f is discontinuous at every point on $[0,1]$.

(Case 1). Let $x \in \mathbb{Q} \cap [0,1]$. Take $\epsilon := 2$. Let $\delta > 0$. Since $[0,1]\backslash\mathbb{Q}$ is dense in $[0,1]$, there exists $y \in [0,1]\backslash\mathbb{Q}$ such that $|x - y| < \delta$. It follows that $|f(x) - f(y)| = |1 - (-1)| = 2 \geq \epsilon$. By the negation of continuity, f is discontinuous at x.

(Case 2). Let $x \in [0,1]\backslash\mathbb{Q}$.The proof follows by interchaning the role of $[0,1] \cap \mathbb{Q}$ and $[0,1]\backslash\mathbb{Q}$ in Case 1 since $[0,1] \cap \mathbb{Q}$ is dense is $[0,1]$ as well.

2 (P.134 Q15)**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at a point $c \in \mathbb{R}$. For all $x \in \mathbb{R}$, define $h(x) := \sup\{f(x), g(x)\}$. Show that the function $h : \mathbb{R} \to \mathbb{R}$ satisfies that

i. $h(x) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$ for all $x \in \mathbb{R}$

ii. Hence, h is continuous at c.

Solution. i. Fix $x \in \mathbb{R}$.

(Case 1). Suppose $f(x) \geq g(x)$. Then $h(x) = f(x)$ and $|f(x) - g(x)| = f(x) - g(x)$. Hecne, we have $h(x) = f(x) = \frac{1}{2}(f(x) + g(x) + f(x) - g(x)) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$

(Case 2). Suppose $g(x) \geq f(x)$. Then $h(x) = g(x)$ and $|f(x) - g(x)| = g(x) - f(x)$. Hecne, we have $h(x) = g(x) = \frac{1}{2}(f(x) + g(x) + g(x) - f(x)) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$

The result follows by combining both cases.

ii. . Since f, g are continuous at c, it suffices to show that continuity at a point is preserved under addition, subtraction and absolute values. The first two are clear. (It follows readily from the definition, or you may use the sequential criteria to deduce them from the cases for sequences). The last one follows from the composition rule: since $f - g$ is continuous at c and $|\cdot|$ is continuous everywhere (proved in HW6), $f(c) - g(c)$ in particular, we have $|f - g| = |\cdot| \circ (f - g)$ is continuous at c.