MATH 2050A - HW 6 - Solutions

(Commonly missed steps are in purple)

We would be using the following results.

Lemma 0.1 (Sequential Criteria: $x \to \infty$)**.** *(Can be used without proofs.) Let A be an unbounded set. Let $f : A \to \mathbb{R}$ be a function. Let $L \in \mathbb{R}$. Then $\lim_{x\to\infty} f(x) = L$ if and only if for all sequences (x_n) in A with $\lim x_n = \infty$, we have $\lim f(x_n) = L$*

Proof. ($\Rightarrow$). Let (x_n) be a sequence in A such that $x_n \to \infty$. Let $\epsilon > 0$. By the assumption, there exists $M > 0$ such that for all $x \in A$ and $x \geq M$, we have $|f(x) - L| < \epsilon$. Furthermore, by the convergence of (x_n), there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \geq M$. Hence, when $n \geq N$, $x_n \geq M$ and $x_n \in A$. Therefore, $|f(x_n) - L| < \epsilon$.
($\Leftarrow$). Suppose not. Then there exists $\epsilon > 0$ such that for all $M \in \mathbb{R}$, there exists $x \geq M$ and $x \in A$ with $|f(x) - L| \geq \epsilon$. Therefore, we can choose a sequence (x_n) in A such that $x_n \geq n$, but $|f(x_n) - L| \geq \epsilon$ for all $n \in \mathbb{N}$.
We proceed to show that (x_n) converges to ∞. Let $M > 0$. By Archimedean Property, there exists $N \in \mathbb{N}$ with $N \geq M$. Hence if $n \geq N$, $x_n \geq n \geq M$. It follows by definition that $\lim x_n = \infty$. Therefore $\lim f(x_n) = L$ by assumption. By order limit property and the fact that limit can commute with absolute values (which is easy to prove), if follows that $0 = |L - L| \geq \epsilon$, which is a contradiction. □

Lemma 0.2. *(Can be used without proofs.) Let (x_n) be a sequence in $\mathbb{R}$. Suppose $\lim x_n = \infty$. Then (x_n) is nonzero eventually . Furthermore, we have $\lim(x_n^{-1}) = 0$.*

Proof. The proof is left as an exercise. □

Solutions

1 (P.123 Q9)**.** Let $a \in \mathbb{R}$.Let $f : (a, \infty) \to \mathbb{R}$ be a function such that $L := \lim_{x\to\infty} xf(x) \in \mathbb{R}$. Show that $\lim_{x\to\infty} f(x) = 0$

Solution. By Lemma 0.1, it suffices to show that $\lim_n f(x_n) = 0$ for all sequence in (a, ∞) such that $x_n \to \infty$. Let (x_n) be a sequence in (a, ∞) such that $x_n \to \infty$. By Lemma 0.1 again, we have $\lim_n x_n f(x_n) = L$.
Note that by Lemma 0.2, (x_n) is eventually nonzero and $\lim_n (x_n)^{-1} = 0$. Therefore we have $f(x_n) = x_n f(x_n)x_n^{-1}$ for sufficiently large n (where x_n are nonzero). Hence by the product limit property, we have $\lim f(x_n) = \lim x_n f(x_n) \lim x_n^{-1} = L \cdot 0 = 0$.

2 (P.123 Q13)**.** Let $a \in \mathbb{R}$. Let f, g be defined on (a, ∞). Suppose $\lim_{x\to\infty} f = L$ and $\lim_{x\to\infty} g = \infty$ where $L \in \mathbb{R}$. Show that $\lim_{x\to\infty} f \circ g = L$.

Solution. Let $\epsilon > 0$. By the convergence of f, there exists $a < M_1 \in \mathbb{R}$ such that $|f(x) - L| < \epsilon$ for all $x \geq M_1$. By the convergence of g, there exists $a < M_2 \in \mathbb{R}$ such that $g(x) \geq M_1$ for all $x \geq M_2$. Now suppose $x \geq M_2$ and $x \in (a, \infty)$. We have $x \geq M_2$. Hence $g(x) \geq M_1$, which implies $|f \circ g(x) - L| < \epsilon$. The result follows from the definition of limits.

3 (P.129 Q4a)**.** Let $x \in \mathbb{R}$. Define $\lfloor x \rfloor$ to be the greatest integer $n \in \mathbb{Z}$ such that $n \leq x$, for example, $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$. We call $x \mapsto \lfloor x \rfloor$ the floor function, which is defined on $\mathbb{R}$.

i Determine the points of continuity of $f(x) = \lfloor x \rfloor$. *(Do not forget to prove your assertion).*

ii **Optional (+1 Bonus):** Show that the floor function is well-defined.

Solution.

i We claim that f is continuous precisely at $\mathbb{R}\backslash\mathbb{Z}$. Since every point in $\mathbb{R}$ is a limit point in $\mathbb{R}$, we proceed to consider functional limits.
First, f is not continuous on $\mathbb{Z}$. Let $n \in \mathbb{Z}$. Let $\epsilon > 0$. Take $\delta := 1/2$. Then for all $0 < n - x < \delta = 1/2$, we have $n - 1 \leq x < n$. Hence, $|f(x) - (n - 1)| = |\lfloor x \rfloor - (n - 1)| = 0 < \epsilon$.
Meanwhile, for all $0 < x - n < \delta = 1/2$, we have $n \leq x < n + 1$. Similarly, $|f(x) - n| = 0 < \epsilon$. Hence, we have $\lim_{x\to n^-} f(x) = n - 1$ but $\lim_{x\to n^+} f(x) = n$. Since the left and right limit do not coincide, by the Characterization of limits using left, right limits, $\lim_{x\to n} f(x)$ does not exists. As every point $x \in \mathbb{R}$ is a limit point of $\mathbb{R}$, we conclude that f is not continuous at n for all $n \in \mathbb{Z}$.
Next we show that f is continuous at $x \in \mathbb{R}\backslash\mathbb{Z}$. Let $r \in \mathbb{R}\backslash\mathbb{Z}$. Then $\lfloor r \rfloor < r < \lfloor r \rfloor + 1$. Let $\epsilon > 0$. Take $\delta := \min\{r - \lfloor r \rfloor, \lfloor r \rfloor + 1 - r\}$. Then for all $x \in \mathbb{R}$ with $|x - r| < \delta$, we have $\lfloor r \rfloor < x < \lfloor r \rfloor + 1$. Hence $\lfloor x \rfloor = \lfloor r \rfloor$. Therefore $|f(x) - f(r)| = |\lfloor x \rfloor - \lfloor r \rfloor| = 0 < \epsilon$. The result follows by definition of continuity.

ii The solution is on the next page.

4 (P.129 Q10)**.** Show that the absolute value function $f(x) = |x|$ is continuous everywhere on $\mathbb{R}$.

Solution. Fix $x \in \mathbb{R}$. Let $\epsilon > 0$. Take $\delta := \epsilon$. Let $y \in \mathbb{R}$. Suppose $|x - y| < \delta$. By the triangle inequality, we have

$$|f(x) - f(y)| = ||x| - |y|| \leq |x - y| < \delta = \epsilon$$

Hence by definition, f is continuous for all $x \in \mathbb{R}$, that is, continuous everywhere on $\mathbb{R}$.

3 (ii).**Optional(+1 Bonus):** Show that the floor function is well-defined.

Solution. To show the floor function is well-defined, we need to show that the floor function is really a function, that is, first there exists a greatest integer $n \in \mathbb{Z}$ with $n \leq x$. Second, such greatest integer is unique. The latter is clear as the maximum of any subset of real numbers is unique. It remains to verify the former.

Method 1: Using the Well-ordered Property.
The *well-orderness* of natural numbers says that any non-empty subset of natural numbers has a minimum. Note that this is taken as an axiom of natural numbers in the textbook.
By a translation, it can be easily generalized to integer setting, which says that any non-empty subset of integers that is bounded below by *integers* has a minimum.
This could be further generalized to the setting of real numbers, which says that any non-empty subset of integers that is bounded below by *real numbers* has a minimum by using the Archimedean Property. Finally, by multiplying with a negative number, we have that *any non-empty subset of integers that is bounded above by real numbers has a maximum.* (Please fill in the gaps, if any, in the above discussion yourself.)
Now we let $x \in \mathbb{R}$ and $A_x := \{n \in \mathbb{Z} : n \leq x\}$. Then by the Archimedean Property , there exists $N \in \mathbb{N}$ such that $-x \leq N$, which imples $-N \leq x$. Since $-N \in \mathbb{Z}$, we can conclude that A_x is a non-empty subset of integers. Following the above discussion, we can conclude that A has a (unique) maximum. This shows that for any real number x there exists a greatest integer less than or equal to x. Thus, the floor function is well-defined.

Method 2: Using the fact that there is no integer between 0 and 1, that is, $0 \leq n \leq 1$ implies $n = 0$ or $n = 1$ if $n \in \mathbb{Z}$.
Let $x \in \mathbb{R}$ and $A_x := \{n \in \mathbb{Z} : n \leq x\}$. By the Archimedean Property as in Method 1, A_x is non-empty and is bounded above by x. Hence by the Axiom of Completeness, $\sup A_x \in \mathbb{R}$. By definition, there exists $n \in A_x$ such that $\sup(A_x) - 1 < n$. Then $\sup(A_x) < n + 1$ which implies that $n + 1 \notin A_x$ but $n \in A_x$. Hence, by definition of A_x, $n + 1 > x \geq n$. We now claim that $\sup(A_x) = n$. Suppose $m \in A_x$ then $m \leq x$. Suppose $m > n$. This would imply $m \geq n + 1$. Otherwise, we have $0 < m - n < 1$. As integer is closed under addition, $m - n$ is an integer, but there is no integer between 0 and 1. Hence we must have $m \geq n + 1$. But then $x \geq m \geq n + 1$, which contradicts the construction of n. Hence $m \leq n$. Therefore $\sup(A_x) \leq n$. As $n \in A_x$, we have $n = \sup(A_x) = \max(A_x) \in A_x$. To conclude, n is the greatest integer such that $n \leq x$.

Appendix

In fact, we do not have to consider the well-orderness of $\mathbb{N}$ an *axiom* for natural numbers. With $\mathbb{Z}$ being the smallest subring of $\mathbb{R}$ and $\mathbb{N}$ being the subset of $\mathbb{Z}$ greater than 0, the well-orderness property of $\mathbb{N}$ *could be deduced* from definition. The following is the related discussion.

We call a subset $Z \subset \mathbb{R}$ a *subring* if $0, 1 \in Z$ and Z is closed under addition, subtraction and multiplication (not necessarily division), for example, the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, the set of dyadic fractions $\{\frac{a}{2^b} \mid a \in \mathbb{Z}, b \in \mathbb{N}\}$, or in general p-adic fractions (try to guess their definitions) are all subrings of real numbers. One can check that there is a smallest subring of real numbers $\mathbb{R}$ (and in fact any arbitrary ring) in the sense that every subring of real numbers will contain that smallest subring by set inclusion. In fact we can *define* integers to be the smallest subring of $\mathbb{R}$. The following is a characterization of well-orderness of subrings of $\mathbb{R}$ as inspired from the above proofs.

Proposition 1.1. *Let $Z \subset \mathbb{R}$ be a subring. Then the following are equivalent:*

1. *Every non-empty bounded below subset of Z has a* ***minimum****.*
2. *If $x \in Z$ and $0 \leq x \leq 1$, then $x = 0$ or $x = 1$.*

Proof. ($\Rightarrow$.) Suppose there exists $x \in Z$ and $0 < x < 1$. Then by product compatibility of order and Z being closed in multiplication, we have $x^n \in Z$ and $0 < x^n < 1$ for all $n \in \mathbb{Z}^+$. But we know that $\lim x^n = 0 = \inf x^n$, whose proof could be relying only on the bounded monotone theorem (hence the axiom of completeness) of $\mathbb{R}$ without assuming $\mathbb{N}$ has any of the two properties in question, but that $1, 2, \ldots \in \mathbb{N}$ and $1 < 2 < \ldots < n < \ldots$ which follows only from order compatibility with addition and multiplication. This implies the bounded below set $\{x^n\} \subset Z$ has no minimum, contradicting the assumption on Z.
($\Leftarrow$.) By definition, $\mathbb{Z}$ is the smallest subring of $\mathbb{R}$. Hence for all $x \in \mathbb{Z}$, $x \in Z$. This implies that $\mathbb{Z}$ has the property in assumption (since Z has). By the above proof in Method 2, we could define the floor function $\lfloor\cdot\rfloor$ with respect to $\mathbb{Z}$. (Note that to prove the Archimedean Property of $\mathbb{Z}^+$, we only need the additive structure of $\mathbb{Z}$ together with the Axiom of Completeness without using the 2 properties in question.) Let $A \subset Z$ be bounded below. Let $\alpha := \inf(A) \in \mathbb{R}$. Using the floor function, we can define $\lfloor\alpha\rfloor \in \mathbb{Z} \subset Z$ such that $\lfloor\alpha\rfloor \leq \alpha < \lfloor\alpha\rfloor + 1$. By definition of inf, there exists $a \in A \subset Z$ such that $\lfloor\alpha\rfloor \leq \alpha \leq a < \lfloor\alpha\rfloor + 1$. By order compatibility with addition, we have $0 \leq a - \lfloor\alpha\rfloor < 1$. Since, $a, \lfloor\alpha\rfloor \in Z$, we have $a - \lfloor\alpha\rfloor \in Z$. By the assumption, $a - \lfloor\alpha\rfloor = 0$ which implies $\lfloor\alpha\rfloor \leq \alpha \leq a = \lfloor\alpha\rfloor$. This follows that $A \ni a = \lfloor\alpha\rfloor = \alpha = \inf(A) = \min(A)$ is the minimum of A. □

Corollary 1.2. *There is a unique subring Z of $\mathbb{R}$ with the property that every non-empty bounded below subset has a minimum.*

Proof. (Uniqueness:) Let Z_1, Z_2 be subrings with the property. Then we claim $Z_1 = Z_2$. Being a subset to such subrings, $\mathbb{Z}$ has the property in question and so the floor function could be defined. Let $x \in Z_1$. Suppose $x \notin Z_2$. Then by the assumption, $\lfloor x\rfloor < x < \lfloor x\rfloor + 1$. However, as $x \in Z_1$, we also have $x = \lfloor x\rfloor$ or $x = \lfloor x\rfloor + 1$. This is a contradiction. Hence it must be that $x \in Z_2$. Therefore, $Z_1 \subset Z_2$. The other direction is similar. Hence, $Z_1 = Z_2$.
(Existence:) We claim $\mathbb{Z}$ is the unique subring having the desired property. Note that being the smallest subring of $\mathbb{R}$, $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\}$. Hence there exists no $x \in \mathbb{Z}$ with $0 < x < 1$ by the symmetry of $\leq$ as a partial ordering. This is because we have the ordering below by order compatibility with both addition and multiplication:

$$\ldots < -n < \ldots < -1 < 0 < 1 < \ldots < n < \ldots$$

□