MATH 2050A - HW 4 - Solutions

We would be using the following results. These can be used without proof.

Lemma 0.1. *Let $x := (x_n)$ be a converging sequence. Then any subsequence of x is convergent and converges to* $\lim_n x_n$*.*

Proof. See Lecture Note Proposition 3.3. □

Lemma 0.2. *[Monotone Convergence Theorem for sequences] Let (x_n) be a sequence of real numbers. Suppose (x_n) is increasing (resp. decreasing) and is bounded above (resp. bounded below), then (x_n) converges. Furthermore, we have $\lim_n x_n = \sup\{x_n\}$ (resp. $\lim_n x_n = \inf\{x_n\}$).*

Proof. See Lecture Note Proposition 2.13 □

Lemma 0.3. *Define $x_n := (1 + 1/n)^n$. Then x_n is increasing and bounded above. We denote $e := \lim_n x_n$ and $e \leq 3$.*

Proof. See Lecture Note Example 2.14 □

Lemma 0.4. *Let $x := (x_n)$ be a sequence. We call a subsequence a tail of x if it is indexed by $K + \mathbb{N}$ for some $K \in \mathbb{N}$, that is, the subsequence is in the form (x_{n+K}) . Then (x_n) converges if and only if some tail $(x_{n+K}), K \in \mathbb{N}$ of (x_n) converges.*

Proof. By now, you should have a sense this statement is true almost by definition of convergence of sequence as sequential convergence concerns only *eventual* behaviour of the sequence. We leave the proof as an Exericse. You may use it without proof. □

Lemma 0.5. *Let $a \geq 0$ and $n \in \mathbb{N}$. Then $a \geq 1$ if and only if $a^n \geq 1$*

Proof. We include the proof here only for completeness. This can be used without proof.
($\Rightarrow$). It follows from order compatibility with product.
($\Leftarrow$). Suppose not. Then we have $a^n \geq 1$, but $a < 1$. By order compatibility with product we have $a^n \leq 1$. (Note the we only have axioms concerning partial orders, $a \geq 0, b \geq c$ imples $ab \geq ac$.) By symmetry of $\leq$, $a^n = 1$. Therefore for all $k \in \mathbb{N}$, we have $a^{nk} = (a^n)^k = 1$. Since subsequences of a convergent sequence converge to the sequential limit, we have $\lim_k a^{nk} = \lim a^k = 0$ as $a < 1$. (Recall that the proof of $\lim_k a^k = 0$ when $0 \leq a < 1$ does not require taking nth roots. Hence, there is no circular reasoning.) However $a^{nk} = 1$ for all $k \in \mathbb{N}$, implying $\lim_k a^{nk} = 1 \neq 0$. Contradiction aries by uniqueness of limit. We must then have $a \geq 1$ □

Remark. Actually, one can deduce from $a^n = 1$ and $a \geq 0$ that $a = 1$ (how?), but we deliberately use sequential technique here.
The result is still true if $\geq$ is replaced by $\leq$.

Remark. Combining both directions yields that $x \mapsto x^q$ is increasing on $\mathbb{R}_{\geq 0}$ for all $q \in \mathbb{Q}$. (How?)

Lemma 0.6. *Let (x_n) be a sequence of non-negative real numbers such that. Let $k \in \mathbb{N}$. Then (x_n) converges if and only if (x_n^k) converges.*

Proof. We include the proof here only for completeness. This can be used without proof.
($\Rightarrow$). The result follows by a finite application of the product rule for limit. We indeed have $\lim_n x_n^k = (\lim_n x_n)^k$
($\Leftarrow$). Note that $x_n^k \geq 0$ by Lemma 0.5. Therefore, $\lim_n x_n^k \geq 0$. Let $L \geq 0$ be such that $L^k = \lim_n x_n^k$.
(Case 1: $L > 0$). Let $\epsilon > 0$. Since (x_n^k) converges, there exists $N \in \mathbb{N}$, such that $|x_n^k - L^k| < \epsilon/L^{k-1}$ as $n \geq N$. Hence as $n \geq N$, we have

$$|x_n - L| = \frac{|x_n^k - L^k|}{x_n^{k-1} + x_n^{k-2}L^1 + \ldots + L^{k-1}} \leq \frac{|x_n^k - L^k|}{L^{k-1}} < \epsilon$$

where the first equality is just some algebraic manipulation. By the $\epsilon - N$ definition for sequences (x_n) converges.
(Case 2: $L = 0$). We leave this as an Exercise. The result follows by combining the two cases. □

Remark. This result with its proof shows that taking limit can commute with taking rational powers for *non-negative* sequences. Of course, taking integer power can commute with limit for *arbitrary* (convergent) real sequences by finite application of the limit product rule. Suitable caution has to be taking with negative powers for sure.

Solutions

1 (P.84 Q4a)**.** Show that $\left(1-(-1)^n+\frac{1}{n}\right)$ is divergent.

Solution. Define $x_n := 1-(-1)^n+1/n$ for all $n \in \mathbb{N}$. Consider the subsequences $(y_n) := (x_{2n})$ and $(z_n) := (x_{2n-1})$; they are subsequences as $n \mapsto 2n$ and $n \mapsto 2n-1$ are strictly increasing. Note that $y_n = 1/n$ for all $n \in \mathbb{N}$. Hence $\lim_n y_n = \lim_n 1/n = 0$. Meanwhile, note that $z_n = 2+1/n$. Hence, $\lim_n z_n = 2+\lim_n 1/n = 2$ by the sum law for limit. Since $0 \neq 2$, (y_n) and (z_n) are two subsequences of (x_n) converging to different limit. The result follows by Lemma 0.1.

2 (P.84 Q7a)**.** Establish the convergence for the following sequence and find its limit.

$$\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)$$

Solution. We show that the sequence converges to e, the natural base.
Define $x_n := (1+1/n)^n$ for all $n \in \mathbb{N}$. By Lemma 0.3, we know that $\lim_n x_n = e$. We consider the subsequence $(y_n) = (x_{n^2})$ as $n \mapsto n^2$ is strictly increasing. Since subsequence of a convergent sequence converges to the sequential limit by Lemma 0.1., we have $\lim_n y_n = \lim_n x_n = e$. The result follows by noting that (y_n) is the sequence in question.

3 (P.84 Q8a)**.** Determine the limit of the following sequence. (It it possible that the limit does not exist).

$$\left((3n)^{\frac{1}{2n}}\right)$$

Solution. We proceed by showing that $\lim_n n^{1/n} = 1$. Define $x_n := n^{1/n}$. We first show that (x_n) is decreasing for sufficiently large n, that is, some tail of (x_n) is decreasing.
By the Archimedean Principle, we pick $K \in \mathbb{N}$ such that $e \leq N$ where e is the natural base (or simply take $K = 3$). Then for all $n \geq K$, we have by Lemma 0.3

$$\left(\frac{x_{n+1}}{x_n}\right)^{n(n+1)} = \left(\frac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}}\right)^{n(n+1)} = \frac{(n+1)^n}{n^{n+1}} = \left(1+\frac{1}{n}\right)^n \frac{1}{n} \leq \frac{e}{n} \leq \frac{e}{N} \leq 1$$

Since taking *rth* root preserves partial order(see Lemma 0.5), taking $n(n+1)$th root on both sides, we have for all $n \geq K$,

$$\frac{x_{n+1}}{x_n} \leq 1$$

that is, the tail (x_{n+K}) is a decreasing sequence. We denote this tail subsequence $(y_n) := (x_{n+K})$. Clearly (y_n) is bounded below since x_n are non-negative for all $n \in \mathbb{N}$. By the monotone convergence theorem for sequences (Lemma 0.2), (y_n) converges and $\lim_n y_n = \inf\{y_n\}$.
We proceed to show that $\inf\{y_n\} = \inf\{x_n\} = 1$. Note that $n \geq 1$ implies $x_n = n^{1/n} \geq 1$ for all $n \in \mathbb{N}$ as *rth* root preserves partial order. So 1 is a lower bound for $\{x_n\}$.
Suppose 1 is not the greatest lower bound, then there exists $r > 1$ such that $r \leq n^{1/n}$ for all $n \in \mathbb{N}$. Hence, $r^n \leq n \implies 1 \leq nr^{-n}$ for all $n \in \mathbb{N}$. By taking limit on both sides, we have $1 \leq 0$. (See Lemma 0.2 in HW3 Solution for why the LHS converges to 0).
Therefore, we can conclude $\inf\{x_n\} = 1$ and we have $1 \leq \inf\{y_n\} \leq \inf\{x_n\} = 1$. So, $\inf\{y_n\} = 1$. Therefore, $\lim_n y_n = \lim_n x_{n+K}$. By Lemma 0.4, we have $\lim_n n^{1/n} = \lim_n x_n = 1$.

Let $(x_n) := (n^{1/n})$. Then the sequence in the question is given by $(x_{3n}^{3/2})$. By considering subsequences and the remark in lemma 0.6, we have $\lim_n(x_{3n}^{3/2}) = (\lim_n x_{3n})^{3/2} = (\lim_n x_n)^{3/2} = 1^{3/2} = 1$.