Chapter 5: Invertible Matrices

5.1 Inverse of a Matrix

Example 5.1.1: Consider the system

$$\begin{array}{rcrcrcr} -7x_1 & - & 6x_2 & - & 12x_3 & = & -33, \\ 5x_1 & + & 5x_2 & + & 7x_3 & = & 24, \\ x_1 & & & + & 4x_3 & = & 5. \end{array}$$

We can represent this system of equations as

$$A\boldsymbol{x} = \boldsymbol{b},$$

where

$$A = \begin{pmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{pmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}.$$

Now, entirely unmotivated, we define a 3×3 matrix B,

$$\begin{pmatrix} -10 & -12 & -9 \\ 13/2 & 8 & 11/2 \\ 5/2 & 3 & 5/2 \end{pmatrix}$$

and note the remarkable fact that

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now apply this computation to the problem of solving the system of equations,

$$\boldsymbol{x} = I_3\boldsymbol{x} = (BA)\boldsymbol{x} = B(A\boldsymbol{x}) = B\boldsymbol{b}.$$

So we have

$$\boldsymbol{x} = B\boldsymbol{b} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}.$$

So with the help and assistance of B we have been able to determine a solution to the system represented by $A\boldsymbol{x} = \boldsymbol{b}$ through judicious use of matrix multiplication. Since the coefficient matrix in this example is nonsingular, there would be a unique solution, no matter what the choice of $\boldsymbol{b}$. The derivation above amplifies this result, since we were *forced* to conclude that $\boldsymbol{x} = B\boldsymbol{b}$ and the solution could not be anything else. You should notice that this argument would hold for any particular choice of $\boldsymbol{b}$. ■

The matrix B of the previous example is called the inverse of A. When A and B are combined via matrix multiplication, the result is the identity matrix, which can be inserted *in front* of $\boldsymbol{x}$ as the first step in finding the solution.

This is entirely analogous to how we might solve a single linear equation with one unknown like $3x = 12$.

$$x = 1x = \left(\frac{1}{3}(3)\right)x = \frac{1}{3}(3x) = \frac{1}{3}(12) = 4.$$

Here we have obtained a solution by employing the *multiplicative inverse* of 3, $3^{-1} = \frac{1}{3}$. This works fine for any scalar multiple of x, except for zero, since zero does not have a multiplicative inverse. Consider separately the two linear equations,

$$0x = 12; \quad 0x = 0.$$

The first has no solutions, while the second has infinitely many solutions. For matrices, it is all just a little more complicated. Some matrices have inverses, some do not.

And when a matrix does have an inverse, just how would we compute it? In other words, just where did that matrix B in the last example come from? Are there other matrices that might have worked just as well?

Definition 5.1.1: Suppose A and B are square matrices of order n such that $AB = I_n$ and $BA = I_n$. Then A is *invertible* and B is the *inverse* of A. Here, we use 'the' not 'an', since we shall show that inverse of a matrix is unique if it exists. In this situation, we write $B = A^{-1}$.

Notice that if B is the inverse of A, then we can just as easily say A is the inverse of B, or A and B are inverses of each other.

From Remark 2.5.9 we considered two matrices: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. We got $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Suppose A has the inverse, say C. Then $AC = I_2$. Hence $B = BI_2 = BAC = \mathcal{O}_2 C = \mathcal{O}_2$ which yields a contradiction.

So, NOT every square matrix has an inverse.

Example 5.1.2: Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}.$$

It is easy to see that $AB = BA = I_2$. So B is the inverse of A. ■

Example 5.1.3: Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -3 & 3 & 6 & -1 & -2 \\ 0 & -2 & -5 & -1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} -3 & 3 & 6 & -1 & -2 \\ 0 & -2 & -5 & -1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} -3 & 3 & 6 & -1 & -2 \\ 0 & -2 & -5 & -1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

So by the definition of inverse matrix, we can say that A is invertible and write $B = A^{-1}$. ■

We will now concern ourselves less with whether or not an inverse of a matrix exists, but instead with how you can find one when it does exist. Later we will have some theorems that allow us to more quickly and easily determine just when a matrix is invertible.

5.2 Computing the Inverse of a Matrix

Theorem 5.2.1: *Suppose* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Then* A *is invertible if and only if* $ad - bc \neq 0$. *When* A *is invertible,*

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof: [⇐] Assume that $ad - bc \neq 0$. Let $B = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$BA = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We get that $A^{-1} = B$.

[⇒]

Example 5.2.1: Consider the matrix in Example 5.1.3

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{pmatrix}.$$

Suppose there is a matrix B such that $AB = I_5$. Then we have

$$A\left(B_{*1}|B_{*2}|B_{*3}|B_{*4}|B_{*5}\right) = (\boldsymbol{e}_1|\boldsymbol{e}_2|\boldsymbol{e}_3|\boldsymbol{e}_4|\boldsymbol{e}_5)$$
$$(AB_{*1}|AB_{*2}|AB_{*3}|AB_{*4}|AB_{*5}) = (\boldsymbol{e}_1|\boldsymbol{e}_2|\boldsymbol{e}_3|\boldsymbol{e}_4|\boldsymbol{e}_5)$$

Equating the matrices column-by-column we have

$$AB_{*1} = \boldsymbol{e}_1, \quad AB_{*2} = \boldsymbol{e}_2, \quad AB_{*3} = \boldsymbol{e}_3, \quad AB_{*4} = \boldsymbol{e}_4, \quad AB_{*5} = \boldsymbol{e}_5.$$

Since the matrix B is what we are trying to compute, we can view each column, B_{*i}, as a column vector of unknowns. Then we have five systems of equations to solve, each with 5 equations in 5 unknowns. Notice that all 5 of these systems have the same coefficient matrix. We will now solve each system.
Row-reduce the augmented matrix of the linear system $AB_{*1} = \boldsymbol{e}_1$

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 1 & 2 & 1 & 1 \\ -2 & -3 & 0 & -5 & -1 & 0 \\ 1 & 1 & 0 & 2 & 1 & 0 \\ -2 & -3 & -1 & -3 & -2 & 0 \\ -1 & -3 & -1 & -3 & 1 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{rrrrr|r} \textcircled{1} & 0 & 0 & 0 & 0 & -3 \\ 0 & \textcircled{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 0 & 1 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 1 \end{array}\right); \quad \text{we get } B_{*1} = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Row-reduce the augmented matrix of the linear system $AB_{*2} = \boldsymbol{e}_2$

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & -5 & -1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 0 \\ -2 & -3 & -1 & -3 & -2 & 0 \\ -1 & -3 & -1 & -3 & 1 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{rrrrr|r} \textcircled{1} & 0 & 0 & 0 & 0 & 3 \\ 0 & \textcircled{1} & 0 & 0 & 0 & -2 \\ 0 & 0 & \textcircled{1} & 0 & 0 & 2 \\ 0 & 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -1 \end{array}\right); \quad \text{we get } B_{*2} = \begin{bmatrix} 3 \\ -2 \\ 2 \\ 0 \\ -1 \end{bmatrix}.$$

Row-reduce the augmented matrix of the linear system $AB_{*3} = \boldsymbol{e}_3$

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & -5 & -1 & 0 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ -2 & -3 & -1 & -3 & -2 & 0 \\ -1 & -3 & -1 & -3 & 1 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{rrrrr|r} \textcircled{1} & 0 & 0 & 0 & 0 & 6 \\ 0 & \textcircled{1} & 0 & 0 & 0 & -5 \\ 0 & 0 & \textcircled{1} & 0 & 0 & 4 \\ 0 & 0 & 0 & \textcircled{1} & 0 & 1 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -2 \end{array}\right); \quad \text{we get } B_{*3} = \begin{bmatrix} 6 \\ -5 \\ 4 \\ 1 \\ -2 \end{bmatrix}.$$

Row-reduce the augmented matrix of the linear system $AB_{*4} = \boldsymbol{e}_4$

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & -5 & -1 & 0 \\ 1 & 1 & 0 & 2 & 1 & 0 \\ -2 & -3 & -1 & -3 & -2 & 1 \\ -1 & -3 & -1 & -3 & 1 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{rrrrr|r} \textcircled{1} & 0 & 0 & 0 & 0 & -1 \\ 0 & \textcircled{1} & 0 & 0 & 0 & -1 \\ 0 & 0 & \textcircled{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 0 & 1 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 0 \end{array}\right); \quad \text{we get } B_{*4} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Row-reduce the augmented matrix of the linear system $AB_{*5} = \boldsymbol{e}_5$

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & -5 & -1 & 0 \\ 1 & 1 & 0 & 2 & 1 & 0 \\ -2 & -3 & -1 & -3 & -2 & 0 \\ -1 & -3 & -1 & -3 & 1 & 1 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{rrrrr|r} \textcircled{1} & 0 & 0 & 0 & 0 & -2 \\ 0 & \textcircled{1} & 0 & 0 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 0 & 0 & -1 \\ 0 & 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 1 \end{array}\right); \quad \text{we get } B_{*5} = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

By this method, we know that $AB = I_5$. We have checked in Example 5.1.3 that $BA = I_5$. So $B = A^{-1}$. ∎

We see that we follows the exact same row operations for each case. We can combine all five cases into one.

$$(A|\boldsymbol{e}_1|\boldsymbol{e}_2|\boldsymbol{e}_3|\boldsymbol{e}_4|\boldsymbol{e}_5) = (A|I_5) \xrightarrow{rref} (I_5|B).$$

Example 5.2.2: Find the inverse of the matrix

$$A = \begin{pmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{pmatrix},$$

which is shown in Example 5.1.1.

Form the augmented matrix

Theorem 5.2.2: *Suppose A is a nonsingular square matrix of order n. Create the $n \times 2n$ matrix $M = (A|I_n)$. Let N be a matrix that is row-equivalent to M and in rref. Finally, let P be the matrix formed from the last n columns of N in order. Then $AP = I_n = PA$. In other word, $P = A^{-1}$.*

5.3 Properties of Matrix Inverses

Theorem 5.3.1: *Suppose the square matrix A has an inverse B. Then it is unique.*

Theorem 5.3.2: *Suppose A and B are invertible. Then*

1. *AB is invertible. Moreover, $(AB)^{-1} = B^{-1}A^{-1}$.*
2. *A^{-1} is invertible and $(A^{-1})^{-1} = A$.*
3. *A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.*
4. *Suppose $c \neq 0$. cA is invertible and $(cA)^{-1} = c^{-1}A^{-1}$.*

Notice that there are some likely theorems that are missing here. For example, it would be tempting to think that $(A+B)^{-1} = A^{-1} + B^{-1}$, but this is false. Can you find a counterexample?

Theorem 5.3.3: *Suppose that $A, B \in M_n$. The product AB is nonsingular if and only if A and B are both nonsingular.*

Proof: [$\Rightarrow$] For this portion of the proof we will form the logically-equivalent contrapositive and prove that statement using two cases.

AB is nonsingular implies A and B are both nonsingular becomes *A or B is singular implies AB is singular.*

Theorem 5.3.4: *Suppose A and B are square matrices of order n such that $AB = I_n$. Then $BA = I_n$.*

The above theorem tells us that if A is nonsingular, then the matrix B guaranteed by Theorem 5.2.2 will be both a *right-inverse* and a *left-inverse* for A. So A is invertible and $A^{-1} = B$.

So if you have a nonsingular matrix A, you can use the procedure described in Theorem 5.2.2 to find an inverse for A. If A is singular, then the procedure in Theorem 5.2.2 will fail as the first n columns of M will not row-reduce to the identity matrix. However, we can say a bit more. When A is singular, then A does not have an inverse (which is very different from saying that the procedure in Theorem 5.2.2 fails to find an inverse). This may feel like we are splitting hairs, but it is important that we do not make unfounded assumptions. These observations motivate the next theorem.

Theorem 5.3.5: *Suppose that $A \in M_n$. Then A is nonsingular if and only if A is invertible.*

So for a square matrix, the properties of having an inverse and of having a trivial null space are one and the same. Update the Theorem 4.3.5 we have

Theorem 5.3.6: *Suppose that $A \in M_n$. The following are equivalent.*

1. *A is nonsingular.*
2. *A is row equivalent to I_n.*
3. *$\mathcal{N}(A) = \{\mathbf{0}_n\}$.*
4. *The linear system $\mathcal{LS}(A, b)$ has a unique solution for every possible choice of $\boldsymbol{b}$.*
5. *A is invertible.*

Theorem 5.3.7: *Suppose that A is nonsingular. Then the unique solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $A^{-1}\boldsymbol{b}$.*

Proof: A is nonsingular implies the system $A\boldsymbol{x} = \boldsymbol{b}$ has unique solution. It is easy to check that $A^{-1}\boldsymbol{b}$ is a solution. □

5.4 Elementary Matrices

Recall that

1 An elementary matrix of type 1 is $(b\mathcal{R}_i)(I) = I + (b-1)E^{i,i}$ for $b \neq 0$

2 An elementary matrix of type 2 is $(b\mathcal{R}_i + \mathcal{R}_j)(I) = I + bE^{j,i}$ for $i \neq j$ and $b \neq 0$ having the form

3 An elementary matrix of type 3 is $(\mathcal{R}_i \leftrightarrow \mathcal{R}_j)(I) = I - E^{i,i} - E^{j,j} + E^{i,j} + E^{j,i}$ for $i \neq j$

Here $E^{i,j} \in M_m(\mathbb{R})$ whose (i,j)-entry is 1 and others are zero, where m fixed, i.e., $[E^{i,j}]_{h,k} = \delta_{hi}\delta_{kj}$.

Corollary 3.1.4: *Suppose $A \in M_{m,n}$. After performing s elementary row operations we obtain B. There are s elementary matrices $E_1, \ldots, E_s$ such that*

$$B = E_s \cdots E_1 A = PA,$$

here $P = E_s \cdots E_1$, a product of elementary matrices.

We shall show that the matrix P is invertible. In order to show this result, we have to induce a useful formula about matrices $E^{i,j}$.

Lemma 5.4.1: *For any integers i, j, h, k, $E^{i,j}E^{h,k} = \delta_{jh}E^{i,k}$.*

Proof: Suppose the matrices are of order m. Then

$$\begin{aligned}[E^{i,j}E^{h,k}]_{x,y} &= \sum_{z=1}^{m}[E^{i,j}]_{x,z}[E^{h,k}]_{z,y} = \sum_{z=1}^{m}\delta_{xi}\delta_{zj}\delta_{zh}\delta_{yk} \\ &= \delta_{jh}\delta_{xi}\delta_{yk} = \delta_{jh}[E^{i,k}]_{x,y}.\end{aligned}$$

Therefore, we have the lemma. $\square$

Proposition 5.4.2: *Elementary matrices are invertible and their inverses are also elementary matrices of the same type.*

Proof: Let E be an elementary matrix.

For the type 1 case, $E = I + (b-1)E^{i,i}$ for $b \neq 0$. Let $C = I + (b^{-1}-1)E^{i,i}$. Then

$$\begin{aligned} EC &= [I + (b-1)E^{i,i}][I + (b^{-1}-1)E^{i,i}] \\ &= I + (b-1)E^{i,i} + (b^{-1}-1)E^{i,i} + (b-1)(b^{-1}-1)E^{i,i}E^{i,i}. \end{aligned}$$

By Lemma 5.4.1, $EC = I + (b + b^{-1} - 2)E^{i,i} + (2 - b - b^{-1})\delta_{ii}E^{i,i} = I$. By Theorem 5.3.4 we get that $C = E^{-1}$.

For the type 2 case, $E = I + bE^{j,i}$ for $i \neq j$ and $b \neq 0$. Let $C = I - bE^{j,i}$. Since $i \neq j$, by Lemma 5.4.1 $E^{j,i}E^{j,i} = \mathcal{O}$. So $EC = CE = I - b^2E^{j,i}E^{j,i} = I$.

For the type 3 case, you can check that $EE = I$, that is E is the inverse of E. It is left to you. $\square$

Remark 5.4.3: Since $(E^{i,j})^t = E^{j,i}$, if E is an elementary matrix then so is E^t.

Combining with Corollary 3.1.4, Proposition 5.4.2 and Theorem 5.3.2 we have

Theorem 5.4.4: *Let $A, B \in M_{m,n}$. Then B is row-equivalent to A if and only if $B = PA$, where P is a product of $m \times m$ elementary matrices. Moreover, such P is invertible.*

5.5 Uniqueness of RREF

By the definition of reduced row echelon form, we have the following two lemmas.

Lemma 5.5.1: *Suppose $C = \begin{bmatrix} A & B \end{bmatrix}$ is in rref. Then A is also in rref.*

Lemma 5.5.2: *Suppose $C = \begin{bmatrix} A \\ B \end{bmatrix}$ is in rref. Then A and B are also in rref.*

Theorem 5.5.3: *Suppose A and B are row-equivalent and are in rref. Let A' and B' be the result of removing the last k columns of A and B, respectively. Then A' and B' are row-equivalent and are in rref.*

Lemma 5.5.4: *If $(H|\boldsymbol{b})$ and $(H|\boldsymbol{c})$ are in rref and are row-equivalent, then $\boldsymbol{b} = \boldsymbol{c}$.*

Theorem 5.5.5: *If two $m \times n$ matrices A and B are in rref and are row-equivalent, then $A = B$.*

Proof: Suppose $A \neq B$. Let k be the least integer so that the k-th column of A does not agree with the k-th column of B. Consider the submatrices

$$\left(\begin{array}{ccc|c} A_{*1} & \cdots & A_{*(k-1)} & A_{*k} \end{array} \right) \quad \text{and} \quad \left(\begin{array}{ccc|c} B_{*1} & \cdots & B_{*(k-1)} & B_{*k} \end{array} \right).$$

By Theorem 5.5.3 the above matrices are in rref and are row-equivalent.

If $k = 1$, then $A_{*1} = \mathbf{0}_m$ or $\boldsymbol{e}_1$. Since A_{*1} is row-equivalent to B_{*1}, $A_{*1} = \mathbf{0}_m$ if and only if $B_{*1} = \mathbf{0}_m$. So if $A_{*1} = \boldsymbol{e}_1$, then B_{*1} is not a zero column. Thus it must a leading column. Hence $B_{*1} = \boldsymbol{e}_1$.

If $k > 1$, then by Lemma 5.5.4 $A_{*k} = B_{*k}$.

For both cases, we have $A_{*k} = B_{*k}$. It is a contradiction. □

By Theorem 5.5.5 the rref of a given matrix A is unique. We use $\text{rref}(A)$ to denote the rref of A. The number of nonzero rows of $\text{rref}(A)$, say r, (i.e, the number of pivots, the number of leading columns) is called the *rank* of A and denoted by $\text{rank}(A)$.

[In other section, rank of A is defined to be the dimension of the column space of A and denoted by $r(A)$. But they are equivalent. It will be mentioned later.]

Theorem 5.5.6: *Let $A \in M_{m,n}$. Then* $\text{rank}(A) \leq \min\{m, n\}$.