Chapter 4: Homogeneous Systems and Nonsingular Matrices

4.1 Homogeneous Systems

Definition 4.1.1: A system of linear equations is *homogeneous* if the vector of constants is the zero vector, i.e.,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & & & \vdots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

Definition 4.1.2: The *homogenous system* corresponding to a linear system $A\boldsymbol{x} = \boldsymbol{b}$ is $A\boldsymbol{x} = \mathbf{0}$. We often say that $A\boldsymbol{x} = \mathbf{0}$ is the *homogenous part* of $A\boldsymbol{x} = \boldsymbol{b}$.

Example 4.1.1: The following is a homogenous system of linear equations:

$$\begin{array}{ccccccccc} x_1 & - & 2x_2 & + & 3x_3 & - & 4x_4 & = & 0 \\ & & x_2 & & & - & x_4 & = & 0 \\ x_1 & + & 3x_2 & - & 5x_3 & + & 5x_4 & = & 0 \end{array}$$

It is the homogenous part of the linear system

$$\begin{array}{ccccccccc} x_1 & - & 2x_2 & + & 3x_3 & - & 4x_4 & = & 1 \\ & & x_2 & & & - & x_4 & = & 2 \\ x_1 & + & 3x_2 & - & 5x_3 & + & 5x_4 & = & 3 \end{array}$$

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Theorem 4.1.3: *Suppose that a system of linear equations is homogeneous. Then the system is consistent. In fact* $\mathbf{0}$ *is a solution, i.e.,* $x_1 = x_2 = \cdots = x_n = 0$ *is a solution. Such solution is called a* trivial solution.

Example 4.1.2:

$$\begin{array}{ccccccc} -7x_1 & - & 6x_2 & - & 12x_3 & = & 0 \\ 5x_1 & + & 5x_2 & + & 7x_3 & = & 0 \\ x_1 & & & + & 4x_3 & = & 0 \end{array}$$

The reduced row echelon form of the augmented matrix is

$$\left(\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \end{array}\right)$$

It has $n - r = 3 - 3 = 0$ free variables. Hence it has only the trivial solution. ■

Notice that when we do row operations on the augmented matrix of a homogeneous system of linear equations the last column of the matrix is all zeros. Any row operation will convert zeros to zeros and thus, the final column of the matrix in reduced row-echelon form will also be all zeros. So we may ignore the last column of the augmented matrix, i.e., we only consider the coefficient matrix for a homogeneous system.

Example 4.1.3:

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & 2x_3 & = & 0 \\ 2x_1 & + & x_2 & + & x_3 & = & 0 \\ x_1 & + & x_2 & & & = & 0 \end{array}$$

The reduced row echelon form of the coefficient matrix is

Example 4.1.4:

$$\begin{array}{rcrcrcrcl} 2x_1 & + & x_2 & + & 7x_3 & - & 7x_4 & = & 0 \\ -3x_1 & + & 4x_2 & - & 5x_3 & - & 6x_4 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & - & 5x_4 & = & 0 \end{array}$$

Applying elementary row operations on the coefficient matrix:

Theorem 4.1.4: *Suppose that a homogeneous system of linear equations has m equations and n unknowns with $n > m$. Then the system has infinitely many solutions.*

Proof: By Theorem 4.1.3, the system is consistent. By Theorem 3.4.3, the system has infinitely many solutions. □

If $n = m$, then we can have a unique solution or infinitely many solutions (see Examples 4.1.2 and 4.1.3).

4.2 Null Space of a Matrix

Definition 4.2.1: The *null space* of a matrix A, denoted by $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $A\boldsymbol{x} = \mathbf{0}$.

Example 4.2.1: Suppose

$$A = \begin{pmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{pmatrix}.$$

Then

$$\boldsymbol{x} = \begin{pmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{y} = \begin{pmatrix} -4 \\ 1 \\ -3 \\ -2 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

are in $\mathcal{N}(A)$ as $A\boldsymbol{x} = \mathbf{0}_4$ and $A\boldsymbol{y} = \mathbf{0}_4$. However, the vector

$$\boldsymbol{z} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \notin \mathcal{N}(A) \text{ as } A\boldsymbol{z} = \begin{pmatrix} -17 \\ 16 \\ -16 \\ 73 \end{pmatrix} \neq \mathbf{0}_4.$$

Example 4.2.2: Let us compute the null space of

$$A = \begin{pmatrix} 2 & -1 & 7 & -3 & -8 \\ 1 & 0 & 2 & 4 & 9 \\ 2 & 2 & -2 & -1 & 8 \end{pmatrix}.$$

Translating Definition 4.2.1, we simply desire to solve the homogeneous system $A\boldsymbol{x} = \mathbf{0}$.

For saving space, we write

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -2x_3 - x_5 & 3x_3 - 4x_5 & x_3 & -2x_5 & x_5 \end{bmatrix}^T \;\middle|\; x_3, x_5 \in \mathbb{R} \right\} \text{ or}$$

$$\mathcal{N}(A) = \{(-2x_3 - x_5, 3x_3 - 4x_5, x_3, -2x_5, x_5) \mid x_3, x_5 \in \mathbb{R}\}.$$

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Example 4.2.3: Compute the null space of

$$C = \begin{pmatrix} -4 & 6 & 1 \\ -1 & 4 & 1 \\ 5 & 6 & 7 \\ 4 & 7 & 1 \end{pmatrix}.$$

4.3 Nonsingular Matrices

In this section we specialize further and consider matrices with equal numbers of rows and columns, which when considered as coefficient matrices lead to systems with equal numbers of equations and unknowns.

Definition 4.3.1: Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $A\boldsymbol{x} = \mathbf{0}$ is $\{\mathbf{0}\}$, in other words, the system has *only* the trivial solution. Then we say that A is a *nonsingular* matrix. Otherwise we say A is a *singular* matrix.

Example 4.3.1: Let A be the matrix in Example 4.1.3. Since the system of linear equations $A\boldsymbol{x} = \mathbf{0}$ has nontrivial solutions.

Hence A is singular.

Let

$$B = \begin{pmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{pmatrix}.$$

By Example 4.1.2, the system of linear equations $B\boldsymbol{x} = \mathbf{0}$ has only trivial solutions. So it is nonsingular. ■

Theorem 4.3.2: *Suppose that A is a square matrix. A is nonsingular if and only if $\text{rref}(A)$ is the identity matrix.*

Corollary 4.3.3: *Suppose that A is a square matrix. Then A is nonsingular if and only if $\mathcal{N}(A) = \{\mathbf{0}\}$.*

Proof: $\mathcal{N}(A) = \{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\}$. By definition A is nonsingular if and only if $\{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\} = \{\mathbf{0}\}$. Hence we have the corollary. □

Theorem 4.3.4: *Suppose that $A \in M_n$. A is nonsingular if and only if the system $\mathcal{LS}(A, \boldsymbol{b})$ has a unique solution for every choice of the constant vector $\boldsymbol{b}$.*

Theorem 4.3.5: *Suppose that $A \in M_n$. The following are equivalent.*

1. *A is nonsingular.*

2. *A is row equivalent to I_n.*

3. *$\mathcal{N}(A) = \{\mathbf{0}_n\}$.*

4. *The linear system $\mathcal{LS}(A, \boldsymbol{b})$ has a unique solution for every possible choice of $\boldsymbol{b}$.*

Proof: The statement that A is nonsingular is equivalent to each of the subsequent statements by Theorem 4.3.2, Corollary 4.3.3 and Theorem 4.3.4. $\square$

The next theorem tells us that in order to find all of the solutions to a linear system of equations, it is sufficient to find just one solution, and then find all of the solutions to the corresponding homogeneous system. This explains part of our interest in the null space, the set of all solutions to a homogeneous system.

Theorem 4.3.6: *Suppose that $\boldsymbol{p}$ is one solution to the linear system of equations $\mathcal{LS}(A, \boldsymbol{b})$. Then $\boldsymbol{y}$ is a solution to $\mathcal{LS}(A, \boldsymbol{b})$ if and only if $\boldsymbol{y} = \boldsymbol{p} + \boldsymbol{z}$ for some vector $\boldsymbol{z} \in \mathcal{N}(A)$, i.e.,*

1. *If $\boldsymbol{y}$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$, then $\boldsymbol{y} - \boldsymbol{p} \in \mathcal{N}(A)$.*

2. *If $\boldsymbol{z} \in \mathcal{N}(A)$, then $\boldsymbol{p} + \boldsymbol{z}$ is a solution of $A\boldsymbol{x} = \boldsymbol{b}$.*

In other words, there is a one-to-one correspondence between

$$\text{solution set of } A\boldsymbol{x} = \boldsymbol{b} \longleftrightarrow \mathcal{N}(A),$$

through

$$\boldsymbol{y} \mapsto \boldsymbol{y} - \boldsymbol{p},$$
$$\boldsymbol{p} + \boldsymbol{z} \leftarrow\!\shortmid \boldsymbol{z}.$$

Proof:

So, we may write the solution set of $A\boldsymbol{x} = \boldsymbol{b}$ as

$$\{\boldsymbol{z} + \boldsymbol{p} \mid \boldsymbol{z} \in \mathcal{N}(A)\},$$

where $\boldsymbol{p}$ is a solution of $A\boldsymbol{x} = \boldsymbol{b}$. The solution $\boldsymbol{p}$ is called a *particular solution* of $A\boldsymbol{x} = \boldsymbol{b}$. The above set is often denoted by $\mathcal{N}(A) + \boldsymbol{p} = \boldsymbol{p} + \mathcal{N}(A)$.

Example 4.3.2: Consider the system

$$
\begin{array}{rcrcrcrcr}
2x_1 & + & x_2 & + & 7x_3 & - & 7x_4 & = & 8, \\
-3x_1 & + & 4x_2 & - & 5x_3 & - & 6x_4 & = & -12, \\
x_1 & + & x_2 & + & 4x_3 & - & 5x_4 & = & 4.
\end{array}
$$