Chapter 10: Eigenvalues and Eigenvectors

10.1 Eigenvalues and Eigenvectors of a Matrix

Definition 10.1.1: Suppose that A is a square matrix of order n, $\alpha \neq \mathbf{0}$ is a vector in $\mathbb{R}^n$, and λ is a scalar in $\mathbb{R}$. We say that α is an *eigenvector* of A with *eigenvalue* λ if

$$A\alpha = \lambda\alpha.$$

Example 10.1.1: Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Let

$$\alpha = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \ \gamma = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Then

$$A\alpha = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} = 4\alpha, \ A\beta = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1\beta, \ A\gamma = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1\gamma.$$

So α is an eigenvector of A with eigenvalue 4, β is an eigenvector of A with eigenvalue 1, γ is an eigenvector of A with eigenvalue 1.
Now let $\eta = 100\alpha$. Then

$$A\eta = 100A\alpha = 400\alpha = 4\eta.$$

So η is an eigenvector of A with eigenvalue 4.
Next let $\theta = \beta + \gamma$, then

$$A\theta = A\beta + A\gamma = \beta + \gamma = 1\theta.$$

So θ is an eigenvector of A with eigenvalue 1.
Let $\phi = \alpha + \beta = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. Then

$$A\phi = 4\alpha + \beta = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

is not a multiple of ϕ. So ϕ is not an eigenvector. This shows that sum of eigenvectors may not be an eigenvector. ■

Example 10.1.2: Consider the matrix

$$A = \begin{pmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{pmatrix}$$

and the vectors

$$\alpha = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 5 \end{pmatrix}, \quad \beta = \begin{pmatrix} -3 \\ 4 \\ -10 \\ 4 \end{pmatrix}, \quad \gamma = \begin{pmatrix} -3 \\ 7 \\ 0 \\ 8 \end{pmatrix}, \quad \theta = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 0 \end{pmatrix}.$$

Then

$$A\alpha = \begin{pmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 8 \\ 20 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ -1 \\ 2 \\ 5 \end{pmatrix} = 4\alpha.$$

So α is an eigenvector of A with eigenvalue $\lambda = 4$.

Also,

$$A\beta = \begin{pmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{pmatrix} \begin{pmatrix} -3 \\ 4 \\ -10 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} -3 \\ 4 \\ -10 \\ 4 \end{pmatrix} = 0\beta.$$

Thus, β is an eigenvector of A with eigenvalue $\lambda = 0$.

Also,

$$A\gamma = \begin{pmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{pmatrix} \begin{pmatrix} -3 \\ 7 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} -6 \\ 14 \\ 0 \\ 16 \end{pmatrix} = 2 \begin{pmatrix} -3 \\ 7 \\ 0 \\ 8 \end{pmatrix} = 2\gamma.$$

So γ is an eigenvector of A with eigenvalue $\lambda = 2$.

Also,

$$A\theta = \begin{pmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 8 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \\ 4 \\ 0 \end{pmatrix} = 2\theta.$$

Thus, θ is an eigenvector of A with eigenvalue $\lambda = 2$. ■

So we have demonstrated four eigenvectors of A. Are there more? Yes, any nonzero scalar multiple of an eigenvector is again an eigenvector. In this example, set $\eta = 30\alpha$. Then

$$A\eta = A(30\alpha) = 30A\alpha = 30(4\alpha) = 4(30\alpha) = 4\eta.$$

Now, η is also an eigenvector of A for the same eigenvalue, $\lambda = 4$.

The vectors γ and θ are both eigenvectors of A for the same eigenvalue $\lambda = 2$, yet this is not as simple as the two vectors just being scalar multiples of each other (they are not). Look what happens when we add them together, to form $\sigma = \gamma + \theta$, and multiply by A,

$$A\sigma = A(\gamma + \theta) = A\gamma + A\theta = 2\gamma + 2\theta = 2(\gamma + \theta) = 2\sigma.$$

So that σ is also an eigenvector of A for the eigenvalue $\lambda = 2$.

It would appear that the set of eigenvectors that are associated with a fixed eigenvalue is closed under the vector space operations of $\mathbb{R}^n$.

The vector β is an eigenvector of A for the eigenvalue $\lambda = 0$. So $A\beta = 0\beta = \mathbf{0}$. But this also means that $\beta \in \mathcal{N}(A)$. There would appear to be a connection here also.

10.2 Existence of Eigenvalues and Eigenvectors

Suppose $A \in M_n$.

$$A\alpha = \lambda\alpha \iff A\alpha - \lambda I_n \alpha = \mathbf{0} \iff (A - \lambda I_n)\,\alpha = \mathbf{0}.$$

So, for an eigenvalue λ and associated eigenvector $\alpha \neq \mathbf{0}$, the vector α will be a nonzero element of the null space of $A - \lambda I_n$, while the matrix $A - \lambda I_n$ will be singular and therefore have zero determinant. These ideas motivate the following definition and example.

Definition 10.2.1: Suppose that $A \in M_n$. The *characteristic polynomial* of A is the polynomial $p_A\,(x)$ defined by

$$p_A\,(x) = \det(A - xI_n).$$

Example 10.2.1: Consider

$$F = \begin{pmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{pmatrix}.$$

Then

$$\begin{aligned}
p_F\,(x) = \det(F - xI_3) &= \begin{vmatrix} -13-x & -8 & -4 \\ 12 & 7-x & 4 \\ 24 & 16 & 7-x \end{vmatrix} \\
&= (-13-x)\begin{vmatrix} 7-x & 4 \\ 16 & 7-x \end{vmatrix} + (-8)(-1)\begin{vmatrix} 12 & 4 \\ 24 & 7-x \end{vmatrix} + (-4)\begin{vmatrix} 12 & 7-x \\ 24 & 16 \end{vmatrix} \\
&= (-13-x)((7-x)(7-x)-4(16)) + (-8)(-1)(12(7-x)-4(24)) + (-4)(12(16)-(7-x)(24)) \\
&= 3 + 5x + x^2 - x^3 = -(x-3)(x+1)^2.
\end{aligned}$$

■

Theorem 10.2.2: *Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A\,(\lambda) = 0$.*

Proof:

Theorem 10.2.3: *Suppose that $A \in M_n$. The characteristic polynomial of A, $p_A\,(x)$, has degree n.*

Proof: The following briefly explain why the theorem is true. It is not a rigorous proof.

$$p_A\,(x) = \begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{vmatrix}$$

The determinant is a sum of products of entries of $A - xI_n$, all the products has degree $\leq n-1$ except the product of the diagonals

$$(a_{11} - x)(a_{22} - x) \cdots (a_{nn} - x),$$

which has degree n.

Remark: We can also see that the leading coefficient is $(-1)^n$. $\square$

Example 10.2.2: In Example 10.2.1, we found the characteristic polynomial of F to be $p_F(x) = -(x-3)(x+1)^2$.

We can find all of its roots easily. They are $x = 3$ and $x = -1$. By the previous theorem, $\lambda = 3$ and $\lambda = -1$ are both eigenvalues of F, and these are the only eigenvalues of F. We have found them all. $\blacksquare$

Definition 10.2.4: Suppose that A is a square matrix and λ is an eigenvalue of A. The *eigenspace* of A for λ, $\mathcal{E}_A(\lambda)$, is the set of all the eigenvectors of A for λ, together with the inclusion of the zero vector.

Theorem 10.2.5: *Suppose $A \in M_n$ and λ is an eigenvalue of A. Then*

$$\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n).$$

Proof: First, notice that $\mathbf{0} \in \mathcal{E}_A(\lambda)$ and $\mathbf{0} \in \mathcal{N}(A - \lambda I_n)$. Now consider any nonzero vector $\alpha \in \mathbb{R}^n$,

$$\begin{aligned} \alpha \in \mathcal{E}_A(\lambda) &\iff A\alpha = \lambda\alpha \iff A\alpha - \lambda\alpha = \mathbf{0} \\ &\iff A\alpha - \lambda I_n\alpha = \mathbf{0} \iff (A - \lambda I_n)\,\alpha = \mathbf{0} \iff \alpha \in \mathcal{N}(A - \lambda I_n). \end{aligned}$$

$\square$

Corollary 10.2.6: *Suppose $A \in M_n$ and λ is an eigenvalue of A. The eigenspace $\mathcal{E}_A(\lambda)$ is a subspace of the vector space $\mathbb{R}^n$.*

Proof: $\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$ is a subspace of $\mathbb{R}^n$ (Theorem 6.2.4). $\square$

Example 10.2.3: Examples 10.2.1 and 10.2.2 described the characteristic polynomial and eigenvalues of the matrix F

We will now take each eigenvalue in turn and compute its eigenspace. To do this, we row-reduce the matrix $F - \lambda I_3$ in order to determine solutions to the homogeneous system $(F - \lambda I_3)\boldsymbol{x} = \mathbf{0}$ and then express the eigenspace as the null space of $F - \lambda I_3$. Then we can write the null space as the span of a basis.

10.3 More Examples

Example 10.3.1: Consider the matrix

$$B = \begin{pmatrix} -2 & 1 & -2 & -4 \\ 12 & 1 & 4 & 9 \\ 6 & 5 & -2 & -4 \\ 3 & -4 & 5 & 10 \end{pmatrix}.$$

Then

$$p_B(x) = 8 - 20x + 18x^2 - 7x^3 + x^4 = (x-1)(x-2)^3$$

So the eigenvalues are $\lambda = 1,\ 2$.

Computing eigenvectors,

$$\lambda = 1, \qquad B - 1I_4 = \begin{pmatrix} -3 & 1 & -2 & -4 \\ 12 & 0 & 4 & 9 \\ 6 & 5 & -3 & -4 \\ 3 & -4 & 5 & 9 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_B(1) = \mathcal{N}(B - I_4) = \left\langle \begin{pmatrix} -\frac{1}{3} \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} -1 \\ 3 \\ 3 \\ 0 \end{pmatrix} \right\rangle.$$

$$\lambda = 2, \qquad B - 2I_4 = \begin{pmatrix} -4 & 1 & -2 & -4 \\ 12 & -1 & 4 & 9 \\ 6 & 5 & -4 & -4 \\ 3 & -4 & 5 & 8 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_B(2) = \mathcal{N}(B - 2I_4) = \left\langle \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} -1 \\ 2 \\ -1 \\ 2 \end{pmatrix} \right\rangle.$$

You may see that $\dim \mathcal{E}_B(2) = 1 = \dim \mathcal{E}_B(1)$. But the whole space $\mathbb{R}^4$ is of dimension 4. ■

Example 10.3.2: Consider the matrix

$$C = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Then

$$p_C(x) = -3 + 4x + 2x^2 - 4x^3 + x^4 = (x-3)(x-1)^2(x+1).$$

So the eigenvalues are $\lambda = 3,\ 1,\ -1$.

Computing eigenvectors,

$$\lambda = 3, \qquad C - 3I_4 = \begin{pmatrix} -2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_C(3) = \mathcal{N}(C - 3I_4) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$

$$\lambda = 1, \qquad C - 1I_4 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_C(1) = \mathcal{N}(C - I_4) = \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle.$$

$$\lambda = -1, \qquad C + 1I_4 = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_C(-1) = \mathcal{N}(C + I_4) = \left\langle \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$

You may see that $\dim \mathcal{E}_C(-1) = 1 = \dim \mathcal{E}_C(3)$ and $\dim \mathcal{E}_C(1) = 2$. The sum of these dimension is equal to the dimension of the whole space $\mathbb{R}^4$. ■

Example 10.3.3: Consider the matrix

$$E = \begin{pmatrix} 29 & 14 & 2 & 6 & -9 \\ -47 & -22 & -1 & -11 & 13 \\ 19 & 10 & 5 & 4 & -8 \\ -19 & -10 & -3 & -2 & 8 \\ 7 & 4 & 3 & 1 & -3 \end{pmatrix}.$$

Then

$$p_E(x) = -16 + 16x + 8x^2 - 16x^3 + 7x^4 - x^5 = -(x-2)^4(x+1).$$

So the eigenvalues are $\lambda = 2,\ -1$.

Computing eigenvectors,

$$\lambda = 2, \qquad E - 2I_5 = \begin{pmatrix} 27 & 14 & 2 & 6 & -9 \\ -47 & -24 & -1 & -11 & 13 \\ 19 & 10 & 3 & 4 & -8 \\ -19 & -10 & -3 & -4 & 8 \\ 7 & 4 & 3 & 1 & -5 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_E(2) = \mathcal{N}(E - 2I_5) = \left\langle \begin{pmatrix} -1 \\ \frac{3}{2} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} -2 \\ 3 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} \right\rangle.$$

$$\lambda = -1, \qquad E + 1I_5 = \begin{pmatrix} 30 & 14 & 2 & 6 & -9 \\ -47 & -21 & -1 & -11 & 13 \\ 19 & 10 & 6 & 4 & -8 \\ -19 & -10 & -3 & -1 & 8 \\ 7 & 4 & 3 & 1 & -2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_E(-1) = \mathcal{N}(E + I_5) = \left\langle \begin{pmatrix} -2 \\ 4 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle.$$

■

Example 10.3.4: Consider the matrix

$$H = \begin{pmatrix} 15 & 18 & -8 & 6 & -5 \\ 5 & 3 & 1 & -1 & -3 \\ 0 & -4 & 5 & -4 & -2 \\ -43 & -46 & 17 & -14 & 15 \\ 26 & 30 & -12 & 8 & -10 \end{pmatrix}.$$

Then

$$p_H(x) = -6x + x^2 + 7x^3 - x^4 - x^5 = x(x-2)(x-1)(x+1)(x+3)$$

So the eigenvalues are $\lambda = 2,\ 1,\ 0,\ -1,\ -3$.

Computing eigenvectors,

$$\lambda = 2, \qquad H - 2I_5 = \begin{pmatrix} 13 & 18 & -8 & 6 & -5 \\ 5 & 1 & 1 & -1 & -3 \\ 0 & -4 & 3 & -4 & -2 \\ -43 & -46 & 17 & -16 & 15 \\ 26 & 30 & -12 & 8 & -12 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_H(2) = \mathcal{N}(H - 2I_5) = \left\langle \begin{pmatrix} 1 \\ -1 \\ -2 \\ -1 \\ 1 \end{pmatrix} \right\rangle.$$

$$\lambda = 1, \qquad H - 1I_5 = \begin{pmatrix} 14 & 18 & -8 & 6 & -5 \\ 5 & 2 & 1 & -1 & -3 \\ 0 & -4 & 4 & -4 & -2 \\ -43 & -46 & 17 & -15 & 15 \\ 26 & 30 & -12 & 8 & -11 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_H(1) = \mathcal{N}(H - I_5) = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ -1 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \\ 2 \end{pmatrix} \right\rangle.$$

$$\lambda = 0, \qquad H - 0I_5 = \begin{pmatrix} 15 & 18 & -8 & 6 & -5 \\ 5 & 3 & 1 & -1 & -3 \\ 0 & -4 & 5 & -4 & -2 \\ -43 & -46 & 17 & -14 & 15 \\ 26 & 30 & -12 & 8 & -10 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_H(0) = \mathcal{N}(H - 0I_5) = \mathcal{N}(H) = \left\langle \begin{pmatrix} -1 \\ 2 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

$$\lambda = -1, \qquad H + 1I_5 = \begin{pmatrix} 16 & 18 & -8 & 6 & -5 \\ 5 & 4 & 1 & -1 & -3 \\ 0 & -4 & 6 & -4 & -2 \\ -43 & -46 & 17 & -13 & 15 \\ 26 & 30 & -12 & 8 & -9 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_H(-1) = \mathcal{N}(H + I_5) = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 2 \end{pmatrix} \right\rangle.$$

$$\lambda = -3, \qquad H + 3I_5 = \begin{pmatrix} 18 & 18 & -8 & 6 & -5 \\ 5 & 6 & 1 & -1 & -3 \\ 0 & -4 & 8 & -4 & -2 \\ -43 & -46 & 17 & -11 & 15 \\ 26 & 30 & -12 & 8 & -7 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}_H(-3) = \mathcal{N}(H + 3I_5) = \left\langle \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -1 \\ -2 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} -2 \\ 1 \\ 2 \\ 4 \\ -2 \end{pmatrix} \right\rangle.$$

■

10.4 Similar Matrices

Definition 10.4.1: Suppose A and B are two square matrices of order n. A and B are *similar* if there exists an invertible (a nonsingular) matrix of order n, S, such that $A = S^{-1}BS$. We will also say A is similar to B via S. Finally, we will refer to $S^{-1}BS$ as a *similarity transformation* when we want to emphasize the way S changes B.

Example 10.4.1: Let

$$B = \begin{pmatrix} -4 & 1 & -3 & -2 & 2 \\ 1 & 2 & -1 & 3 & -2 \\ -4 & 1 & 3 & 2 & 2 \\ -3 & 4 & -2 & -1 & -3 \\ 3 & 1 & -1 & 1 & -4 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 1 & 3 & -1 & 1 & 1 \\ -2 & -3 & 3 & 1 & -2 \\ 1 & 3 & -1 & 2 & 1 \end{pmatrix}.$$

Check that S is invertible. Compute

$$\begin{aligned} A &= S^{-1}BS \\ &= \begin{pmatrix} 10 & 1 & 0 & 2 & -5 \\ -1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & 0 & 1 \\ -4 & -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -4 & 1 & -3 & -2 & 2 \\ 1 & 2 & -1 & 3 & -2 \\ -4 & 1 & 3 & 2 & 2 \\ -3 & 4 & -2 & -1 & -3 \\ 3 & 1 & -1 & 1 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 1 & 3 & -1 & 1 & 1 \\ -2 & -3 & 3 & 1 & -2 \\ 1 & 3 & -1 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -10 & -27 & -29 & -80 & -25 \\ -2 & 6 & 6 & 10 & -2 \\ -3 & 11 & -9 & -14 & -9 \\ -1 & -13 & 0 & -10 & -1 \\ 11 & 35 & 6 & 49 & 19 \end{pmatrix}. \end{aligned}$$

So by this construction, we know that A and B are similar. ■

Theorem 10.4.2: *Suppose $A, B, C \in M_n$. Then*

1. *A is similar to A. (Reflexive)*
2. *If A is similar to B, then B is similar to A. (Symmetric)*
3. *If A is similar to B and B is similar to C, then A is similar to C. (Transitive)*

Proof:

Theorem 10.4.3: *Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is, $p_A(x) = p_B(x)$.*

Proof: Let n denote the order of A and B. Since A and B are similar, there exists a nonsingular matrix S, such that $A = S^{-1}BS$. Then

Example 10.4.2: Show that

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}.$$

are not similar.

Answer:

Example 10.4.3: Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to obtain that

$$p_A(x) = p_B(x) = 1 - 2x + x^2 = (x-1)^2.$$

So A and B have equal characteristic polynomials.

If the converse of Theorem 10.4.3 were true, then A and B would be similar. Suppose this is the case. There is a nonsingular matrix S such that $A = S^{-1}BS$. Then

$$A = S^{-1}BS = S^{-1}I_2S = S^{-1}S = I_2.$$

Clearly $A \neq I_2$ and this contradiction tells us that the converse of Theorem 10.4.3 is false. ■

10.5 Diagonalizable

Good things happen when a matrix is similar to a diagonal matrix. For example, the eigenvalues of the matrix are the entries on the diagonal of the diagonal matrix. And it can be a much simpler matter to compute high powers of the matrix. Diagonalizable matrices are also of interest in more abstract settings. Here are the relevant definitions, then our main theorem for this section.

Diagonal matrix was mentioned in Chapter 2. Now we recall the definition and make it more precisely.

Definition 10.5.1: Suppose that $A \in M_n$. Then A is a *diagonal matrix* if $[A]_{ij} = 0$ whenever $i \neq j$, i.e.,

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

It is also denoted by $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Definition 10.5.2: Suppose $A \in M_n(\mathbb{R})$. A is *diagonalizable* (over $\mathbb{R}$) if A is similar to a diagonal matrix, i.e., there exists an invertible matrix $S \in M_n(\mathbb{R})$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$S^{-1}AS = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Example 10.5.1: Let

$$B = \begin{pmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{pmatrix} \text{ and } S = \begin{pmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

It can be checked that S is invertible. Now

$$\begin{aligned} S^{-1}BS &= \begin{pmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -1 & -1 \\ 2 & 3 & 1 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \end{aligned}$$

So B is diagonalizable. ■

There are two questions:

1. How can we know that a square matrix is diagonalizable?
2. How do we find the invertible matrix S such that the matrix is similar to a diagonal matrix via S?

Example 10.5.2: Consider the matrix

$$F = \begin{pmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{pmatrix}$$

in Example 10.2.3. F's eigenvalues and eigenspaces are

$$\lambda = 3, \qquad \mathcal{E}_F(3) = \left\langle \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} \right\rangle$$

$$\lambda = -1, \qquad \mathcal{E}_F(-1) = \left\langle \begin{pmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

Define the matrix S to be the 3×3 matrix whose columns are the three basis vectors in the eigenspaces for F,

$$S = \begin{pmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Check that S is nonsingular (row-reduces to the identity matrix, or has a nonzero determinant).
Remark: After we introduce Theorem 10.5.5, you do not need to check that S is nonsingular.

Check:

$$\begin{aligned} S^{-1}FS &= \begin{pmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 4 & 2 \\ -3 & -1 & -1 \\ -6 & -4 & -1 \end{pmatrix} \begin{pmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

■

The three columns of S form a linearly independent set. This is a characterization of diagonalization (see Theorem 10.5.3 below). Furthermore, the construction in the proof of Theorem 10.5.3 tells us that $S^{-1}FS = \operatorname{diag}(3, -1, -1)$.

Theorem 10.5.3: *Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set $\mathscr{B}$ that contains n eigenvectors of A.*

Proof: ($\Rightarrow$) Suppose A is diagonalizable. There exists an invertible matrix S, real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$S^{-1}AS = \operatorname{diag}(\lambda_1, \ldots, \lambda_n). \tag{10.1}$$

Let $\mathscr{B}$ be the set of columns of S. Since S is invertible (nonsingular), the columns of S are linearly independent.

From (10.1) we have

$$AS = S\operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

$$\begin{bmatrix} AS_{*1} & AS_{*2} & \cdots & AS_{*n} \end{bmatrix} = \begin{bmatrix} \lambda_1 S_{*1} & \lambda_2 S_{*2} & \cdots & \lambda_n S_{*n} \end{bmatrix}.$$

Hence for $1 \le i \le n$

$$AS_{*i} = \lambda_i S_{*i}.$$

Obviously $S_{*i} \neq \mathbf{0}$ since S is nonsingular. So S_{*i} is an eigenvector associate with the eigenvalue λ_i. Hence $\mathscr{B}$ is a linearly independent set consisting of eigenvectors of A.

($\Leftarrow$) Suppose $\mathscr{B} = \{\alpha_1, \ldots, \alpha_n\}$ is a linearly independent set consisting of eigenvectors of A with eigenvalues $\lambda_1, \ldots, \lambda_n$, i.e., $A\alpha_i = \lambda_i\alpha_i$ for $i = 1, \ldots, n$. Let $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Let

$$S = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}.$$

Since $\mathscr{B}$ is linearly independent, S is invertible.

Similar to the above computation

$$AS = \begin{bmatrix} A\alpha_1 & A\alpha_2 & \cdots & A\alpha_n \end{bmatrix} = \begin{bmatrix} \lambda_1\alpha_1 & \lambda_2\alpha_2 & \cdots & \lambda_n\alpha_n \end{bmatrix} = SD.$$

So

$$S^{-1}AS = S^{-1}SD = D.$$

Therefore A is diagonalizable. $\square$

Remark 10.5.4: Notice that the proof is constructive. To diagonalize a matrix, we need only locate n linearly independent eigenvectors. We can construct a nonsingular matrix S using the eigenvectors as columns. So that $S^{-1}AS$ is a diagonal matrix D. The entries on the diagonal of D will be the eigenvalues of the eigenvectors used to create S, *in the same order* as the eigenvectors appear in S.

Theorem 10.5.5: *Suppose $A \in M_n$. Suppose $\lambda_1, \ldots, \lambda_k$ are all the distinct eigenvalues of A. Then A is diagonalizable if and only if*

$$\sum_{i=1}^{k} \dim \mathcal{E}_A(\lambda_i) = \dim \mathcal{E}_A(\lambda_1) + \cdots + \dim \mathcal{E}_A(\lambda_k) = n. \tag{10.2}$$

Furthermore, if the above condition is valid for A. Let $\mathscr{B}_i = \{\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \ldots, \alpha_{id_i}\}$, denote a basis for the eigenspace of λ_i, $\mathcal{E}_A(\lambda_i)$, for $1 \le i \le k$ and $d_i = \dim \mathcal{E}_A(\lambda_i)$. Then

$$\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_k$$

is a set of linearly independent eigenvectors for A with size n.
Moreover, let S be a square matrix with column i of S is the i-th vector of the set $\mathscr{B}$, i.e.,

$$S = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1d_1} | & \alpha_{21} & \cdots & \alpha_{2d_2} | & \cdots\cdots & |\alpha_{k1} & \cdots & \alpha_{kd_k} \end{bmatrix}.$$

Then

$$S^{-1}AS = \operatorname{diag}(\underbrace{\lambda_1, \ldots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \ldots, \lambda_2}_{d_2}, \ldots, \underbrace{\lambda_k, \ldots, \lambda_k}_{d_k}).$$

The proof is omitted here, since it relates to some concept of subspaces which are not included in this course.

Remark 10.5.6: (10.2) can be rewritten as

$$\sum_{i=1}^{k} n(A - \lambda_i I_n) = n(A - \lambda_1 I_n) + \cdots + n(A - \lambda_k I_n) = n$$

or

$$\sum_{i=1}^{k} (n - r(A - \lambda_i I_n)) = (n - r(A - \lambda_1 I_n)) + \cdots + (n - r(A - \lambda_k I_n)) = n.$$

Theorem 10.5.7: *Suppose $A \in M_n$ with n distinct eigenvalues. Then A is diagonalizable.*

The proof is omitted here, since it relates to some concept of subspaces which are not included in this course.

Example 10.5.3: Determine if the matrix B in Example 10.3.1 is diagonalizable.

Answer:

Example 10.5.4: Determine if the matrix C in Example 10.3.2 is diagonalizable.
Answer:

Example 10.5.5: Determine if the matrix H in Example 10.3.4 is diagonalizable.
Answer: Since $p_H(x) = x(x-2)(x-1)(x+1)(x+3)$ has 5 distinct eigenvalues, by Theorem 10.5.7, H is diagonalizable. ■

Example 10.5.6: Diagonalize C in Example 10.3.2.

Answer: From Example 10.5.4 we know that C is diagonalizable. From Example 10.3.2 we know that
$\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ is basis of $\mathcal{E}_C(3)$,
$\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}\right\}$ is a basis of $\mathcal{E}_C(1)$ and
$\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}\right\}$ is a basis of $\mathcal{E}_C(-1)$.

By Theorem 10.5.5, let

$$S = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Then $S^{-1}CS = \mathrm{diag}(3, 1, 1, -1)$. ■

Remark: Unlike Example 10.5.2, we do not need to check that S is invertible. This is guaranteed by Theorem 10.5.5.

Example 10.5.7: Diagonalize H in Example 10.3.4.
By Example 10.5.5, H is diagonalizable. From Example 10.3.4 let

$$S = \begin{pmatrix} 1 & 1 & -1 & 1 & -2 \\ -1 & 0 & 2 & 0 & 1 \\ -2 & -1 & 2 & 0 & 2 \\ -1 & -2 & 0 & -1 & 4 \\ 1 & 2 & 1 & 2 & -2 \end{pmatrix}.$$

Then, by Theorem 10.5.5 we have

$$S^{-1}HS = \operatorname{diag}(2, 1, 0, -1, -3).$$

■

Example 10.5.8: Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $p_A(x) = x^2 + 1$. $x^2 + 1$ does not have any root in $\mathbb{R}$. So A is not diagonalizable (over $\mathbb{R}$). But, if we extend our number system to the set of complex numbers, $\mathbb{C}$, then it is another story. A is diagonalizable over $\mathbb{C}$.

Actually, by letting $S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$, we have

$$S^{-1}AS = \frac{-i}{2}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

where $i^2 = -1$. ■

In this course, we only restrict on $\mathbb{R}$. In practice, for example, to solve some problems in Physics, to solve differential equations, it is very helpful when we adopt the complex numbers.

Example 10.5.9: Determine if

$$J = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad L = \begin{pmatrix} -8 & 6 & 6 \\ -9 & 7 & 6 \\ -9 & 6 & 7 \end{pmatrix}$$

are similar. If they are similar, then find R such that $R^{-1}JR = L$.
Answer: The characteristic polynomials

$$p_J(x) = -(-4 + x)(-1 + x)^2 = p_L(x).$$

(If the characteristic polynomials are different, J and L are not similar, end of the story.)

The distinct eigenvalues of J and L are $\lambda_1 = 4$ and $\lambda_2 = 1$.

Now $\operatorname{rref}(J - 4I_3) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\operatorname{rref}(J - I_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. By solving the linear systems we have

$\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ is a basis for $\mathcal{E}_J(4)$ and $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$ is a basis for $\mathcal{E}_J(1)$.

By Theorem 10.5.5, we can take

$$S = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then

$$S^{-1}JS = \mathrm{diag}(4, 1, 1).$$

Follow a similar method, we can show that L is diagonalizable (fill in the detail). Let

$$Q = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

We have

$$Q^{-1}LQ = \mathrm{diag}(4, 1, 1).$$

10.6 Power of Matrices

Suppose s is a positive integers, recall

$$A^s = \overbrace{A \cdots A}^{s} \text{ and } A^0 = I.$$

Powers of a diagonal matrix are easy to compute, and when a matrix is diagonalizable, it is almost as easy. Suppose A is similar to a diagonal matrix $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. That is, there is an invertible matrix S

$$S^{-1}AS = D.$$

Then

$$A = SDS^{-1}.$$

$$A^s = \underbrace{SDS^{-1}SDS^{-1} \cdots SDS^{-1}}_{s} = S \underbrace{D \cdots D}_{s} S^{-1} = SD^sS^{-1}$$

$$= S\,\mathrm{diag}(\lambda_1^s, \ldots, \lambda_n^s)S^{-1}.$$

Example 10.6.1: Suppose that

$$A = \begin{pmatrix} 19 & 0 & 6 & 13 \\ -33 & -1 & -9 & -21 \\ 21 & -4 & 12 & 21 \\ -36 & 2 & -14 & -28 \end{pmatrix}$$

and we wish to compute A^{20}. Normally this would require 19 matrix multiplications, but since A is diagonalizable, we can simplify the computations substantially.

Answer: First, we diagonalize A by finding its eigenvalues and the corresponding eigenvectors. We get

$$S = \begin{pmatrix} 1 & -1 & 2 & -1 \\ -2 & 3 & -3 & 3 \\ 1 & 1 & 3 & 3 \\ -2 & 1 & -4 & 0 \end{pmatrix}$$

and find

$$\begin{aligned} D &= S^{-1}AS \\ &= \begin{pmatrix} -6 & 1 & -3 & -6 \\ 0 & 2 & -2 & -3 \\ 3 & 0 & 1 & 2 \\ -1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 19 & 0 & 6 & 13 \\ -33 & -1 & -9 & -21 \\ 21 & -4 & 12 & 21 \\ -36 & 2 & -14 & -28 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 & -1 \\ -2 & 3 & -3 & 3 \\ 1 & 1 & 3 & 3 \\ -2 & 1 & -4 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A^{20} = SD^{20}S^{-1} &= S \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{20} S^{-1} \\ &= S \begin{pmatrix} (-1)^{20} & 0 & 0 & 0 \\ 0 & (0)^{20} & 0 & 0 \\ 0 & 0 & (2)^{20} & 0 \\ 0 & 0 & 0 & (1)^{20} \end{pmatrix} S^{-1} \\ &= \begin{pmatrix} 1 & -1 & 2 & -1 \\ -2 & 3 & -3 & 3 \\ 1 & 1 & 3 & 3 \\ -2 & 1 & -4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1048576 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -6 & 1 & -3 & -6 \\ 0 & 2 & -2 & -3 \\ 3 & 0 & 1 & 2 \\ -1 & -1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 6291451 & 2 & 2097148 & 4194297 \\ -9437175 & -5 & -3145719 & -6291441 \\ 9437175 & -2 & 3145728 & 6291453 \\ -12582900 & -2 & -4194298 & -8388596 \end{pmatrix}. \end{aligned}$$