Chapter 1: System of Linear Equations – Introduction and Technique

1.1 Geometric Interpretation of Linear Equations

In secondary school, there is a problem: "Find the intersection point of two given straight lines."

We introduce the xy-coordinates for the plane. So each point in the plane is represented uniquely by an order pair (x, y), say. So, the plane is also called an xy-plane. A straight line in the plane is a part of the plane, which is called a 'subset' of the plane mathematically.

We can show that each point lying in a straight line satisfies a linear equation,
for examples, $2x - 3y = -5$ and $x + y = 5$.

1.1.1 Sets

So let us introduce some basic concepts in Mathematics.

Any 'well-defined' collection of objects is called a *set*. The objects which make up a set will be called *elements* or *members* of the set.

If an object x is an element of a set A, then we write $x \in A$, in which case we say that x *belongs* to A, x *is in* A or A *contains* x, otherwise we write $x \notin A$.

In saying that A is a *well-defined collection of objects* if given any object x, then either $x \in A$ or $x \notin A$ but not both.

A set A containing finite elements is called a *finite set*, otherwise, is an *infinite set*.

Suppose A is a finite set. We may denote the set by putting its complete membership list between a pair of braces: For example

$$A = \{1, 2, 3, 4\}.$$

But for infinite set or finite set containing a huge number of elements, it is impossible to list all its elements. So we use the *set builder* to describe a set:

$$A = \{x \mid P(x)\} \text{ or } \{x \ : \ P(x)\}.$$

Here x is only a symbol. $P(x)$ is a compound statement of x, which is called the *property* of x. Sometimes we use $f(x)$ (a formula about x) instead of x at the symbolic part. For example,

$$A = \{n \mid n \text{ is an integer between 1 and 4}\} = \{1, 2, 3, 4\},$$

$$C = \{n^2 - n + 41 \mid n \text{ is an integer}\}.$$

Definition 1.1.1: Suppose A and B are sets. B is a *subset* of A if $x \in B$ then $x \in A$. In this case, we write $B \subseteq A$ or $A \supseteq B$. In addition, if $B \neq A$, then we write $B \subset A$ or $A \supset B$.

Proposition 1.1.2: *Suppose A and B are sets. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

Thus $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 2, 4, 4\}$ are equal.

Suppose A and B are sets.

The *union* and the *intersection* of A and B are defined as follows:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\},$$

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

The *relative complement* $A \setminus B$ is defined by

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

For a fixed discussion or study, it is carried on within some context. So, the collection of all objects in a particular context or theory is called a *universal set*. We always denote this universal set by $\mathscr{U}$. $\mathscr{U} \setminus B$ is denoted by B^c and called the *complement* of B. We may write

$$B^c = \{x \mid x \notin B\},$$

because it is understood that all elements are in $\mathscr{U}$.

A set containing nothing is called the *empty set*. It will be written as $\{\ \}$ or $\varnothing$. The symbol is not a Greek phi (ϕ), it is borrowed from the Norwegian alphabet 'O'. We regard $\varnothing$ as a subset of any set S because we regard the statement "$x \in \varnothing$ implies $x \in S$" as logically true in a vacuous sense.

Laws of algebra of sets:

1.	$(A^c)^c$	$=$	A	double complementation
2a.	$A \cup B$	$=$	$B \cup A$	commutative laws
b.	$A \cap B$	$=$	$B \cap A$	
3a.	$(A \cup B) \cup C$	$=$	$A \cup (B \cup C)$	associate laws
b.	$(A \cap B) \cap C$	$=$	$A \cap (B \cap C)$	
4a.	$A \cup (B \cap C)$	$=$	$(A \cup B) \cap (A \cup C)]$	distributive laws
b.	$A \cap (B \cup C)$	$=$	$(A \cap B) \cup (A \cap C)]$	
5a.	$A \cup A$	$=$	A	idempotent laws
b.	$A \cap A$	$=$	A	
6a.	$A \cup \varnothing$	$=$	A	identity laws
b.	$A \cap \mathscr{U}$	$=$	A	
c.	$A \cup \mathscr{U}$	$=$	$\mathscr{U}$	universal bound laws
d.	$A \cap \varnothing$	$=$	$\varnothing$	
7a.	$A \cup A^c$	$=$	$\mathscr{U}$	negation laws
b.	$A \cap A^c$	$=$	$\varnothing$	
8a.	$(A \cup B)^c$	$=$	$A^c \cap B^c$	DeMorgan laws
b.	$(A \cap B)^c$	$=$	$A^c \cup B^c$	
9a.	U^c	$=$	$\varnothing$	
b.	$\varnothing^c$	$=$	$\mathscr{U}$	

1.1.2 Geometric Interpretation

Now, let us go back to the finding intersection point problem.

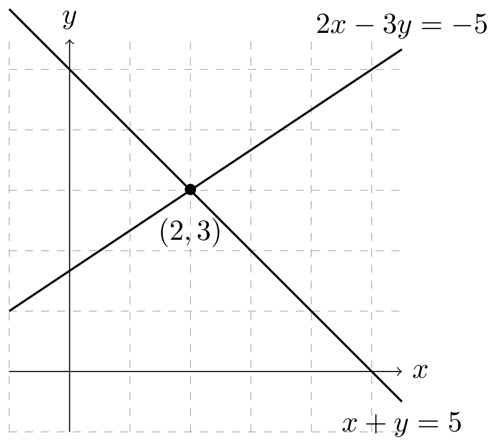
The set of real numbers is denoted by $\mathbb{R}$ (or $\mathbf{R}$). So the set of all points in the xy-plane is $\{(x, y) \mid x, y \in \mathbb{R}\}$ which is denoted by $\mathbb{R}^2$. This is the universal set of the current consideration.

Now, the straight line $2x - 3y = -5$ is the subset $L_1 = \{(x, y) \in \mathbb{R}^2 \mid 2x - 3y = -5\}$. Similarly, the straight line $x + y = 5$ is the subset $L_2 = \{(x, y) \in \mathbb{R}^2 \mid x + y = 5\}$.

So we often say that a linear equation is represented by a straight line and vice versa.

Example 1.1.1: Find the intersection point of $2x - 3y = -5$ and $x + y = 5$.

For the line L_1 that corresponds to $2x - 3y = -5$, it passes through the point $(0, 5/3)$ (take $x = 0$ and one gets $y = 5/3$) and another the point $(-5/2, 0)$. Hence it is determined. Similarly, the second straight line L_2 is the one passes through the points $(0, 5)$ and $(5, 0)$. So we have the following figure:



To find the intersection point of the lines L_1 and L_2, it is equivalent to find $L_1 \cap L_2$, the intersection of two sets. That means, we want to find a point that satisfies both equations. We will write as

$$\begin{cases} 2x - 3y = -5, \\ x + y = 5. \end{cases}$$

and say that to solve the *system of linear equations*. ■

Example 1.1.2: Consider the equations (straight lines).

$$\begin{cases} 2x - 3y = -5, \\ 2x - 3y = 0. \end{cases}$$

We get

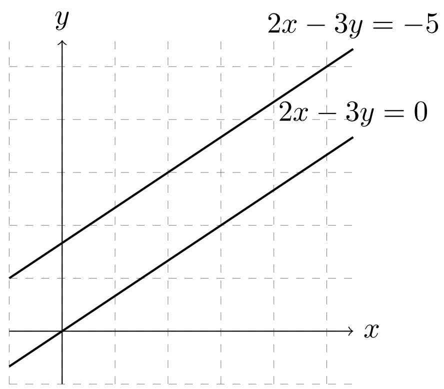


Figure 1: Parallel lines

The corresponding lines are parallel to each other. Thus, and there is no intersections. ■

Example 1.1.3: How about the equations: Consider the equations (straight lines).

$$\begin{cases} 2x - 3y = -5, \\ x - 1.5y = -2.5. \end{cases}$$

There are infinite intersection points, since the two lines coincide and their intersection is just the whole line. ■

Example 1.1.4: Solve the equations:

$$\begin{cases} 2x + 3y + 3z = 3, \\ 2x - y + 0z = 0. \end{cases}$$

There are three variables and we work in 3-dimensional space $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$. So, the universal set of this problem is $\mathbb{R}^3$.

Each (linear) equation corresponds to a plane. For instance, the ones above correspond to the blue and red planes in Figure 2. Their intersection is the thick straight line, where the points on it can be expressed as

$$\left\{ \left(\frac{3-3t}{8}, \frac{3-3t}{4}, t \right) \middle| \; t \in \mathbb{R} \right\}.$$

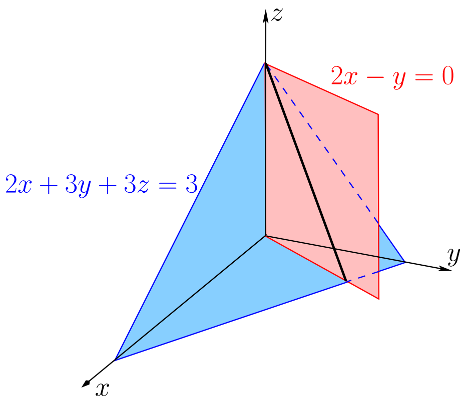


Figure 2: Intersection of planes.

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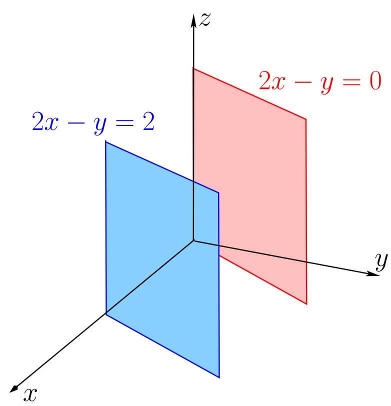
Example 1.1.5: The planes have no intersection since they are parallel.

$$\begin{cases} 2x - y + 0z = 2, \\ 2x - y + 0z = 0. \end{cases}$$



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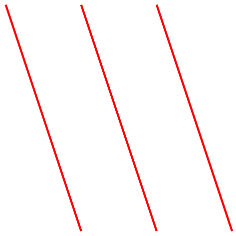
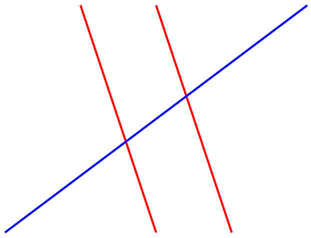
Example 1.1.6 (Singular Case)**:** Suppose that there are three (linear) equations with three variables that corresponds to three distinguish planes. If they do not share a unique intersection, then either they share no intersections or they intersection is a line.

More precisely, there are four cases.

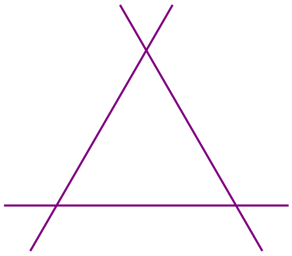
① All three planes are parallel to each other. We shall illustrate this by the following picture.



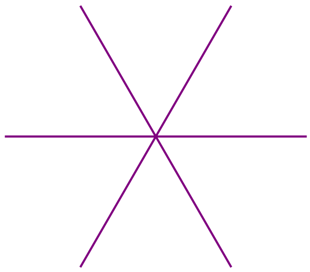
② Only two planes are parallel to each other.



③ The intersection of each pair of planes is a line and three such lines are parallel to each other.



④ Their intersection is a line.



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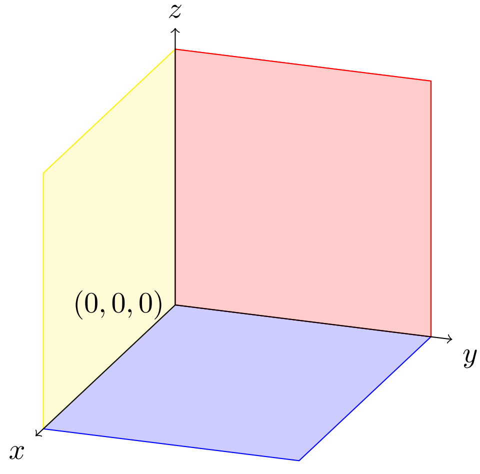
The intersection of three planes is a point, for example, the xy-plane, yz-plane and zx-plane intersect at $(0,0,0)$.



Remark 1.1.3 (Higher dimensions)**:** How does this row picture extend into n dimensions? The n equations will contain n unknowns. The first equation still determines a plane. It is no longer a two dimensional plane in 3-space; somehow it has dimension $n-1$. It must be flat and extremely thin within n-dimensional space, although it would look solid to us.

If time is the fourth dimension, then the plane $t = 0$ cuts through four-dimensional space and produces the three-dimensional universe we live in (or rather, the universe as it was at $t = 0$). Another plane is $z = 0$, which is also three-dimensional; it is the ordinary $x - y$ plane taken over all time. Those three-dimensional planes will intersect! They share the ordinary $x - y$ plane at $t = 0$. We are down to two dimensions, and the next plane leaves a line. Finally a fourth plane leaves a single point. It is the intersection point of 4 planes in 4 dimensions, and it solves the 4 underlying equations.

I will be in trouble if that example from relativity goes any further. The point is that linear algebra can operate with any number of equations. The first equation produces an $(n-1)$-dimensional plane in n dimensions. The second plane intersects it (we hope) in a smaller set of dimension $n-2$. Assuming all goes well, every new plane (every new equation) reduces the dimension by one. At the end, when all n planes are accounted for, the intersection has dimension zero. It is a point. It lies on all the planes, and its coordinates satisfy all n equations. It is the solution!

So, we conclude that to solve a system of equations there are three cases:

1. exactly one solution;
2. no solution;
3. infinite many solutions.

1.2 Algebraic Method

1.2.1 Substitution

Example 1.2.1: Solve

$$\begin{cases} 3x + 4y = 2, & (1.1) \\ 4x + 5y = 3. & (1.2) \end{cases}$$

Use (1.2), we can solve for y in terms of x:

$$y = \frac{3}{5} - \frac{4}{5}x. \tag{1.3}$$

Substitution this into (1.1):

$$3x + 4\left(\frac{3}{5} - \frac{4}{5}x\right) = 2$$
$$\Rightarrow \frac{12}{5} - \frac{x}{5} = 2$$
$$\Rightarrow x = 2.$$

Substituting $x = 2$ into (1.3), we can solve for y:

$$y = \frac{3}{5} - \frac{4}{5} \times 2 = -1.$$

So the solution is $x = 2$, $y = -1$.

Remark : There are other ways to use substitution, for example

1. Solve x by (1.2) in terms of y and substitute it into (1.1)

2. Solve y by (1.1) in terms of x and substitute it into (1.2)
3. Solve x by (1.1) in terms of y and substitute it into (1.2)

But you **cannot** get the solution by

1. Solve y by (1.2) in terms of x and substitute it into (1.2) (No substitution back to the original equation)
2. Solve y by (1.1) in terms of x and substitute it into (1.1)
3. Solve x by (1.2) in terms of y and substitute it into (1.2)
4. Solve x by (1.1) in terms of y and substitute it into (1.1)

■

1.2.2 Elimination

Example 1.2.2: Again, let us solve the system of linear equations from the previous example.

Consider (1.1) $-\frac{3}{4}\times$ (1.2).

$$\begin{array}{rcccccc} & 3x & + & 4y & = & 2 \\ -) & 3x & + & \frac{15}{4}y & = & \frac{9}{4} \\ \hline & & & \frac{1}{4}y & = & -\frac{1}{4} \end{array}$$

Thus $y=-1$. Substituting it into (1.1):

$$3x+4(-1)=2$$

$$x=2.$$

So we obtain the solution $x=2$, $y=-1$.

Remark:

1. The number $\frac{3}{4}$ is so chosen such that the coefficient of x is eliminated.
2. At some point, we still need to use substitution to get the solution.

■

Essentially, elimination and substitution are the same with different presentations. One may try to solve x by (1.2) in terms of y and substitute it into (1.1).

Let us show you one more example in three unknowns to illustrate the above fact.

1.2.1 Substitution

Example 1.2.3: Solve the following system of linear equations

$$\begin{cases} x+2y+2z=4 & (1.4) \\ x+3y+3z=5 & (1.5) \\ 2x+6y+5z=6 & (1.6) \end{cases}$$

Find x in terms of y,z by (1.4) :

$$x=4-2y-2z. \tag{1.7}$$

Substituting (1.7) into (1.5), we obtain

$$(4 - 2y - 2z) + 3y + 3z = 5, \qquad \text{i.e., } y + z = 1.$$

Substituting (1.7) into (1.6), we obtain

$$2(4 - 2y - 2z) + 6y + 5z = 6, \qquad \text{i.e., } 2y + z = -2.$$

The equations are reduced to solving a linear system of equations with two unknowns:

$$\begin{cases} y + z = 1 & (1.8) \\ 2y + z = -2 & (1.9) \end{cases}$$

Solve y in terms of z by (1.8):

$$y = 1 - z. \tag{1.10}$$

Then substitute $y = 1 - z$ into (1.9):

$$2(1 - z) + z = -2, \qquad \text{i.e., } z = 4.$$

By (1.10)

$$y = 1 - z = -3.$$

Substitute $y = -3$, $z = 4$ into (1.7),

$$x = 4 - 2y - 2z = 4 - 2 \times (-3) - 2 \times 4 = 2.$$

Hence $x = 2, y = -3, z = 4$ is a solution. ■

1.2.2 Elimination

Using substitution all the way to solve linear equations is not the best way. Instead, we can use elimination to simplify the system of linear equations first.

Example 1.2.4: We solve the following system by a sequence of equation operations.

$$\begin{aligned} x + 2y + 2z &= 4 & \text{(E1)} \\ x + 3y + 3z &= 5 & \text{(E2)} \\ 2x + 6y + 5z &= 6 & \text{(E3)} \end{aligned}$$

$(-1)\times$ (E1), add to (E2):

$$\begin{aligned} x + 2y + 2z &= 4 & \text{(E1)} \\ 0x + 1y + 1z &= 1 & \text{(E2)} \\ 2x + 6y + 5z &= 6 & \text{(E3)} \end{aligned}$$

$(-2)\times$ (E1), add to (E3):

$$\begin{aligned} x + 2y + 2z &= 4 & \text{(E1)} \\ 0x + 1y + 1z &= 1 & \text{(E2)} \\ 0x + 2y + 1z &= -2 & \text{(E3)} \end{aligned}$$

$(-2)\times$ (E2), add to (E3):

$$\begin{aligned} x + 2y + 2z &= 4 && \text{(E1)} \\ 0x + 1y + 1z &= 1 && \text{(E2)} \\ 0x + 0y - 1z &= -4 && \text{(E3)} \end{aligned}$$

$(-1)\times$(E3):

$$\begin{aligned} x + 2y + 2z &= 4 \\ 0x + 1y + 1z &= 1 \\ 0x + 0y + 1z &= 4 \end{aligned}$$

which can be written more clearly as

$$\begin{aligned} x + 2y + 2z &= 4 \\ y + z &= 1 \\ z &= 4 \end{aligned}$$

The third equation requires that $z = 4$ to be true.
Making this substitution into equation 2 we arrive at $y = -3$,
and finally, substituting these values of y and z into the first equation, we find that $x = 2$.

Remark: We can add several more eliminations to solve x, y, z without substitution:

$$\begin{aligned} x + 2y + 2z &= 4 && \text{(E1)} \\ 0x + 1y + 1z &= 1 && \text{(E2)} \\ 0x + 0y + 1z &= 4 && \text{(E3)} \end{aligned}$$

$(-1)\times$ (E3), add to (E2) and $(-2)\times$ (E3), add to (E1):

$$\begin{aligned} x + 2y + 0z &= -4 && \text{(E1)} \\ 0x + 1y + 0z &= -3 && \text{(E2)} \\ 0x + 0y + 1z &= 4 && \text{(E3)} \end{aligned}$$

$(-2)\times$ (E2), add to (E1)

$$\begin{aligned} x + 0y + 0z &= 2 \\ 0x + 1y + 0z &= -3 \\ 0x + 0y + 1z &= 4 \end{aligned}$$

So $x = 2, y = -3, z = 4$ is a solution. ■

1.3 More examples

Example 1.3.1:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= 1 && ① \\ 2x_1 - 4x_2 + x_3 &= 0 && ② \\ x_1 + x_2 - 2x_3 &= -1 && ③ \end{aligned}$$

$(-2) \times ① + ②$:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= \ \ 1 \\ 0x_1 + 6x_2 - 5x_3 &= -2 \\ x_1 + x_2 - 2x_3 &= -1 \end{aligned}$$

$(-1) \times ① + ③$:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= \ \ 1 && ① \\ 0x_1 + 6x_2 - 5x_3 &= -2 && ② \\ 0x_1 + 6x_2 - 5x_3 &= -2 && ③ \end{aligned}$$

$(-1) \times ② + ③$:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= \ \ 1 && ① \\ 0x_1 + 6x_2 - 5x_3 &= -2 && ② \\ 0x_1 + 0x_2 + 0x_3 &= \ \ 0 && ③ \end{aligned}$$

$\frac{1}{6} \times ②$:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= \ \ 1 && ① \\ 0x_1 + x_2 - \frac{5}{6}x_3 &= -\frac{1}{3} && ② \\ 0x_1 + 0x_2 + 0x_3 &= \ \ 0 \end{aligned}$$

$5 \times ② + ①$:

$$\begin{aligned} x_1 + 0x_2 - \frac{7}{6}x_3 &= -\frac{2}{3} \\ 0x_1 + x_2 - \frac{5}{6}x_3 &= -\frac{1}{3} \\ 0x_1 + 0x_2 + 0x_3 &= \ \ 0 \end{aligned}$$

We can express x_1, x_2 in terms of x_3:

$$\begin{aligned} x_1 &= -\frac{2}{3} + \frac{7}{6}x_3 \\ x_2 &= -\frac{1}{3} + \frac{5}{6}x_3 \end{aligned}$$

The solution set is

$$\left\{ \left(-\frac{2}{3} + \frac{7}{6}a, -\frac{1}{3} + \frac{5}{6}a, a \right) \,\middle|\, a \in \mathbb{R}. \right\}$$

■

Example 1.3.2:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= \ \ 1 && ① \\ 2x_1 - 4x_2 + x_3 &= \ \ 0 && ② \\ x_1 + x_2 - 2x_3 &= -2 && ③ \end{aligned}$$

$(-2) \times ① + ②$:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= 1 & ① \\ 0x_1 + 6x_2 - 5x_3 &= -2 & ② \\ x_1 + x_2 - 2x_3 &= -2 & ③ \end{aligned}$$

$(-1) \times ① + ③$:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= 1 & ① \\ 0x_1 + 6x_2 - 5x_3 &= -2 & ② \\ 0x_1 + 6x_2 - 5x_3 &= -3 & ③ \end{aligned}$$

$(-1) \times ② + ③$:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= 1 & ① \\ 0x_1 + 6x_2 - 5x_3 &= -2 & ② \\ 0x_1 + 0x_2 + 0x_3 &= -1 & ③ \end{aligned}$$

The last equation, $0 = -1$ has no solution. So the system of linear equations has no solution. ■

Example 1.3.3:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7 & ① \\ x_1 + x_2 + x_3 - x_4 &= 3 & ② \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 & ③ \end{aligned}$$

$(-1) \times ① + ②$:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7 & ① \\ -x_2 + x_3 - 2x_4 &= -4 & ② \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 & ③ \end{aligned}$$

$(-3) \times ① + ③$:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7 & ① \\ -x_2 + x_3 - 2x_4 &= -4 & ② \\ -5x_2 + 5x_3 - 10x_4 &= -20 & ③ \end{aligned}$$

$(-1) \times ②$:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7 & ① \\ x_2 - x_3 + 2x_4 &= 4 & ② \\ -5x_2 + 5x_3 - 10x_4 &= -20 & ③ \end{aligned}$$

$5 \times ② + ③$:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7 & ① \\ x_2 - x_3 + 2x_4 &= 4 & ② \\ 0 &= 0 \end{aligned}$$

$(-2) \times ② + ①$:

$$\begin{aligned} x_1 \quad\quad +2x_3 - 3x_4 &= -1 \\ x_2 - x_3 + 2x_4 &= 4 \\ 0 &= 0 \end{aligned}$$

The last equation $0 = 0$ is always true, so we can ignore it and only consider the first two equations. We can write more clearly as

$$x_1 = -2x_3 + 3x_4 - 1 \qquad ①$$
$$x_2 = x_3 - 2x_4 + 4 \qquad ②$$

We can analyze the second equation without consideration of the variable x_1. It would appear that there is considerable latitude in how we can choose x_2, x_3, x_4 and make this equation true. Let us choose x_3 and x_4 to be **anything** we please, say $x_3 = a$ and $x_4 = b$.

Now we can take these arbitrary values for x_3 and x_4, substitute them in ①, to obtain

$$x_1 = -1 - 2a + 3b$$

Similarly, ② becomes

$$x_2 = 4 + a - 2b$$

So our arbitrary choices of values for x_3 and x_4 (a and b) translate into specific values of x_1 and x_2. Now we can easily and quickly find many more (infinitely more). Suppose we choose $a = 5$ and $b = -2$, then we compute

$$\begin{aligned} x_1 &= -1 - 2(5) + 3(-2) = -17 \\ x_2 &= 4 + 5 - 2(-2) = 13 \end{aligned}$$

and you can verify that $(x_1, x_2, x_3, x_4) = (-17, 13, 5, -2)$ makes all three equations true. The entire solution set is written as

$$\{(-1 - 2a + 3b,\, 4 + a - 2b,\, a,\, b) \mid a, b \in \mathbb{R}\}.$$

■

Example 1.3.4: Solve the following system of linear equations:

$$x_2 + x_3 + 2x_4 + 2x_5 = 2 \qquad ①$$
$$x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 = 4 \qquad ②$$
$$-2x_1 - x_2 - 3x_3 + 3x_4 + x_5 = 3 \qquad ③$$

Swap ① and ②:

$$x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 = 4 \qquad ①$$
$$x_2 + x_3 + 2x_4 + 2x_5 = 2 \qquad ②$$
$$-2x_1 - x_2 - 3x_3 + 3x_4 + x_5 = 3 \qquad ③$$

$2 \times ① + ③$:

$$x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 = 4 \qquad ①$$
$$x_2 + x_3 + 2x_4 + 2x_5 = 2 \qquad ②$$
$$3x_2 + 3x_3 + 7x_4 + 7x_5 = 11 \qquad ③$$

$(-3) \times ② + ③$:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 &= 4 \quad ① \\ x_2 + \ \ x_3 + 2x_4 + 2x_5 &= 2 \quad ② \\ x_4 + \ \ x_5 &= 5 \quad ③ \end{aligned}$$

Now the system of linear equations looks like an "inverted stair" (echelon). We can then solve the system of linear equations by substitution (**a better method well be given later**). By the last equation:

$$x_4 = 5 - x_5.$$

Solve x_2 in terms of other unknowns by ②:

$$\begin{aligned} x_2 &= 2 - x_3 - 2x_4 - 2x_5 \\ &= 2 - x_3 - 2(5 - x_5) - 2x_5 \\ &= -8 - x_3 \end{aligned}$$

Solve x_1 in terms of other unknowns by ①:

$$\begin{aligned} x_1 &= 4 - 2x_2 - 3x_3 - 2x_4 - 3x_5 \\ &= 4 - 2(-8 - x_3) - 3x_3 - 2(5 - x_5) - 3x_5 \\ &= 10 - x_3 - x_5. \end{aligned}$$

x_3, x_5 can be taken as any values. Set $x_3 = a$, $x_5 = b$, the solution set can be given by

$$\{(10 - a - b, -8 - a, a, 5 - b, b) \mid a, b \in \mathbb{R}\}.$$

A better method

Instead of substitution, we use more elimination:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 &= 4 \quad ① \\ x_2 + \ \ x_3 + 2x_4 + 2x_5 &= 2 \quad ② \\ x_4 + \ \ x_5 &= 5 \quad ③ \end{aligned}$$

$(-2) \times ③ + ②$:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 &= 4 \\ x_2 + \ \ x_3 \qquad\qquad &= -8 \\ x_4 + \ \ x_5 &= 5 \end{aligned}$$

$(-2) \times ③ + ①$:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \qquad + x_5 &= -6 \quad ① \\ x_2 + \ \ x_3 \qquad\qquad &= -8 \quad ② \\ x_4 + x_5 &= 5 \quad ③ \end{aligned}$$

$(-2) \times ② + ①$:

$$\begin{aligned} x_1 \qquad + x_3 \qquad + x_5 &= 10 \\ x_2 + x_3 \qquad\qquad &= -8 \\ x_4 + x_5 &= 5 \end{aligned}$$

Notice the following:

1. The system of equations looks like an echelon.
2. The leftmost unknowns in the equations are x_1, x_2 and x_4.
3. Only the first equation has unknown x_1.
4. Only the second equation has unknown x_2.
5. Only the third equation has unknown x_4.

Move x_3, x_5 to another side.

$$\begin{aligned} x_1 &= 10 - x_3 - x_5 \\ x_2 &= -8 - x_3 \\ x_4 &= 5 - x_5 \end{aligned}$$

The right hand sides have x_3, x_5 as unknowns only and x_3, x_5 can be taken as any values. Set $x_3 = a$, $x_5 = b$, the solution set can be given by

$$\{(10 - a - b, -8 - a, a, 5 - b, b) \mid a, b \in \mathbb{R}\}.$$

■

Observations:

1. Some systems of linear equations have exactly one solution.
2. Some systems of linear equations have no solution.
3. Some systems of linear equations have infinite many solutions. The solutions can be expressed in terms of 1 or 2 (or even more) unknowns.

1.4 Formal Definitions

Definition 1.4.1: A *system of linear equations* is a collection of m equations in the variable quantities x_1, x_2, $x_3, \ldots, x_n$ of the form,

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad & \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m \end{aligned}$$

where the values of $a_{i,j}$, b_i and x_j, $1 \le i \le m$, $1 \le j \le n$, are real numbers. All $a_{i,j}$ (or sometimes written as a_{ij}) are called *coefficients*, all x_i are called *unknowns*.

Definition 1.4.2: $(s_1, s_2, \ldots, s_n)$ is a *solution* of a system of linear equations in n unknowns if we substitute s_1 for x_1, s_2 for x_2, s_3 for x_3, $\ldots$, s_n for x_n, then for every equation of the system the left side will equal the right side, i.e., each equation is true simultaneously.

The *solution set* of a linear system of equations is the set which contains every solution to the system, and nothing more.

Example 1.4.1: The following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7 \\ x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

can be rewritten as

$$\begin{aligned} 1x_1 + 2x_2 + 0x_3 + 1x_4 &= 7 \\ 1x_1 + 1x_2 + 1x_3 - 1x_4 &= 3 \\ 3x_1 + 1x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

So it is a system of linear equations, with $n = 4$ unknowns and $m = 3$ equations. Also,

$$\begin{array}{lllll} a_{11} = 1, & a_{12} = 2, & a_{13} = 0, & a_{14} = 1, & b_1 = 7; \\ a_{21} = 1, & a_{22} = 1, & a_{23} = 1, & a_{24} = -1, & b_2 = 3; \\ a_{31} = 3, & a_{32} = 1, & a_{33} = 5, & a_{34} = -7, & b_3 = 1. \end{array}$$

When $x_1 = -2$, $x_2 = 4$, $x_3 = 2$, $x_4 = 1$. They satisfy the above equations. So $(-2, 4, 2, 1)$ is a solution of the system.
In fact, the system of equations has infinite many solutions. The solution set is

$$\{(-1 - 2a + 3b,\ 4 + a - 2b,\ a,\ b) \mid a, b \in \mathbb{R}\}.$$

■

Example 1.4.2:

1. From Example 1.1.1 we see that the following system of linear equation has **only one** solution.

$$\begin{aligned} 2x_1 - 3x_2 &= -5, \\ x_1 + x_2 &= 5. \end{aligned}$$

   The solution set is $\{(2, 3)\}$.

2. From Example 1.1.2 we see the following system of linear equations has **no** solutions.

$$\begin{aligned} 2x_1 - 3x_2 &= -5, \\ 2x_1 - 3x_2 &= 0. \end{aligned}$$

   The solution set is empty ($\varnothing$).

3. From Example 1.1.3 the following system of linear equation has **infinite many** solution.

$$\begin{aligned} 2x_1 - 3x_2 &= -5, \\ x_1 - 1.5x_2 &= -2.5. \end{aligned}$$

   The solution set is $\{(x_1, x_2) = (\frac{2t+5}{3}, t) \mid t \in \mathbb{R}\}$.

■

Theorem 1.4.3: *A system of linear equations can have (1) a unique solution, or (2) infinitely many solutions, or (3) no solutions.*

Remark 1.4.4: The coefficients a_{ij} in our course are real numbers. So, it is impossible for a system of linear equation to have exactly 2 solutions. But, (can be ignored), for higher mathematics, the above theorem is not entirely true. However, in higher mathematics, the coefficients can be something in so called *finite field*.

1.5 Equivalent Systems and Equation Operations

Definition 1.5.1: Two systems of linear equations are *equivalent* if their solution sets are equal.

Definition 1.5.2: Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an *equation operation*:

1. Swap the locations of two equations in the list of equations.
2. Multiply each term of an equation by a nonzero *scalar* (in our course, scalar is a real number).
3. Multiply each term of one equation by a nonzero scalar, and add these terms to a second equation, on both sides of the equality.

Theorem 1.5.3: *If we apply one of the three equation operations of Definition 1.5.2 to a system of linear equations, then the original system and the transformed system are equivalent.*

Proof: (Sketch) Let S be a system of linear equations and S' be another system of linear equations obtained by applying equation operations on S.

Let $(x_1, \ldots, x_n)$ be a solution for S. Because S' is obtained by equation operations on S, it is obvious that $(x_1, \ldots, x_n)$ is also a solution for S'.

Conversely, suppose $(x_1, \ldots, x_n)$ is a solution for S'. Note that the reverse of the equation operations are also equation operations. Reversing the equation operators on S, we can show that S can be obtained by equation operations on S'. So $(x_1, \ldots, x_n)$ is also a solution for S.

Hence S and S' have the same solution set. □

Example 1.5.1: The following two systems equations are equivalent:

$$\begin{cases} 2x_1 + 3x_2 &= 3 \qquad (1.11) \\ x_1 - x_2 &= 4 \qquad (1.12) \end{cases}$$

and

$$\begin{cases} 5x_2 &= -5 \\ x_1 - x_2 &= 4 \end{cases}$$

In fact, the second system of linear equations is obtained by applying operation 3 on (1.11) of the first system ($(-2) \times (1.12) + (1.11)$). ■