MATH2010 Advanced calculus, 2020-21
HOMEWORK 3
Suggested Solution

1. (a) Let $u = tx$ and $v = ty$ and take the derivative with respect to t of the equation $f(tx, ty) = t^n f(x, y)$. By Chain Rule, we have

$$\begin{aligned}\frac{\partial}{\partial t} f(u, v) &= \frac{\partial}{\partial t} t^n f(x, y) \\ \frac{\partial f(u, v)}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f(u, v)}{\partial v}\frac{\partial v}{\partial t} &= nt^{n-1} f(x, y) \\ x\frac{\partial f(u, v)}{\partial u} + y\frac{\partial f(u, v)}{\partial v} &= nt^{n-1} f(x, y)\end{aligned}$$

Now let $t = 1$, and we will have

$$x\frac{\partial f(x, y)}{\partial x} + y\frac{\partial f(x, y)}{\partial y} = nf(x, y)$$

.

(b) Following (a), we take derivative once more.

$$\frac{\partial}{\partial t}(x\frac{\partial f(u, v)}{\partial u} + y\frac{\partial f(u, v)}{\partial v}) = \frac{\partial}{\partial t}(nt^{n-1} f(x, y))$$

We have

$$\begin{aligned}\frac{\partial}{\partial t}(x\frac{\partial f(u, v)}{\partial u}) &= x(\frac{\partial^2 f(u, v)}{\partial^2 u}\frac{\partial u}{\partial t} + \frac{\partial^2 f(u, v)}{\partial u \partial v}\frac{\partial v}{\partial t}) \\ &= x^2\frac{\partial^2 f(u, v)}{\partial^2 u} + xy\frac{\partial^2 f(u, v)}{\partial u \partial v}\end{aligned}$$

And similarly,

$$\begin{aligned}\frac{\partial}{\partial t}(y\frac{\partial f(u, v)}{\partial v}) &= y(\frac{\partial^2 f(u, v)}{\partial v \partial u}\frac{\partial u}{\partial t} + \frac{\partial^2 f(u, v)}{\partial^2 v}\frac{\partial v}{\partial t}) \\ &= xy\frac{\partial^2 f(u, v)}{\partial v \partial u} + y^2\frac{\partial^2 f(u, v)}{\partial^2 v}\end{aligned}$$

Combining these equations, we have

$$x^2\frac{\partial^2 f(u, v)}{\partial^2 u} + 2xy\frac{\partial^2 f(u, v)}{\partial u \partial v} + y^2\frac{\partial^2 f(u, v)}{\partial^2 v} = n(n-1)t^{n-2} f(x, y)$$

The proof is finished by letting $t = 1$.

2. The curve passes the point (0,0,1) exactly when $t = 1$. First calculate the velocity,

$$r'(t) = (\frac{1}{t}, 1 + \ln t, 1), r'(1) = (1, 1, 1)$$

Next we define the function $f(x,y,z) = xz^2 - yz + \cos xy$.
We know that the gradient ∇f is perpendicular to the level sets of f, in particular to the surface $f(x,y,z) = 1$. Hence we calculate

$$\begin{aligned} f_x(x,y,z) &= z^2 - y\sin xy \\ f_y(x,y,z) &= -z - x\sin xy \\ f_z(x,y,z) &= 2xz - y \end{aligned}$$

Hence we have

$$\nabla f(0,0,1) = (f_x, f_y, f_z)(0,0,1) = (1,-1,0)$$

The conclusion holds by noting that the point (0,0,1) lies on the curve and the surface, and that the gradient of f is perpendicular to the velocity of the curve at this point, i.e.

$$< \nabla f(0,0,1), r'(1) >= 0$$

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3. (a) We need to compare the values of f at critical points and boundary points.

$$\nabla f(x,y,z) = (f_x, f_y) = (2x + y - 6, x + 2y).$$

Let $\nabla f(a,b) = 0$. We have $(a,b) = (4,-2)$ and $f(4,-2) = -12$.

Next consider $x = 0$. Then $f(0,y) = y^2$ for $-3 \leq y \leq 3$. Easy to see the candidates for maxima and minima are $f(0,0) = 0$ and $f(0,-3) = f(0,3) = 9$.

Consider $x = 5$. Then $f(5,y) = y^2 + 5y - 5 = (y + \frac{5}{2})^2 - \frac{45}{4}$ for $-3 \leq y \leq 3$. The candidates are $f(5, -\frac{5}{2}) = -\frac{45}{4}$ and $f(5,3) = 19$.

Consider $y = 3$. Then $f(x,3) = x^2 - 3x + 9$ for $0 \leq x \leq 5$. The candidates are $f(\frac{3}{2}, 3) = \frac{27}{4}$ and $f(5,3) = 19$.

Consider $y = -3$. Then $f(x,-3) = x^2 - 9x + 9$ for $0 \leq x \leq 5$. The candidates are $f(\frac{9}{2}, -3) = -\frac{45}{4}$ and $f(0,-3) = 9$.

Comparing all the candidates, we conclude that the absolute maxima is $f(5,3) = 19$ and the absolute minima is $f(4,-2) = -12$.

(b) First calculate the gradient of f, for $x > 0$ and $y > 0$,

$$\nabla f(x,y) = (e^{-(2x+3y)}(-12xy + 6y), e^{-(2x+3y)}(-18xy + 6x))$$

Let $\nabla f(a,b) = 0$. We have $(a,b) = (\frac{1}{2}, \frac{1}{3})$ and $f(\frac{1}{2}, \frac{1}{3}) = e^{-2}$.

When $xy = 0$, we have $f(x,y) = 0$.

Note that $f(x, y_0)$ tends to 0 as x tends to $+\infty$ for any fixed positive y_0. Similarly, $f(x_0, y)$ tends to 0 as y tends to $+\infty$ for any fixed positive x_0.

Hence we can conclude that the absolute maxima is $f(\frac{1}{2}, \frac{1}{3}) = e^{-2}$.
The absolute minima is 0 attained on the axes.