Determinants

Definition: Let $A \in M_{mn}$. Let $1 \le i \le m$, $1 \le j \le n$.

$A(i|j) \in M_{m-1\, n-1}$ is the matrix you get by removing row i & column j from A.

Example: Let $A = \begin{bmatrix} 3 & 2 & 2 & 1 \\ 1 & 0 & 2 & 3 \end{bmatrix}$ $\quad m=2,\ n=4$

$$A(1|1) = \begin{bmatrix} \cancel{3} & \cancel{2} & \cancel{2} & \cancel{1} \\ \cancel{1} & 0 & 2 & 3 \end{bmatrix}$$

$= [0 \quad 2 \quad 3]$

$$= \begin{bmatrix} 0 & 2 & 5 \end{bmatrix}.$$

Let $A = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

this is not accepted notation

$$A(2|2) = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$A(3|2) = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

<u>Definition</u>: The <u>determinant</u> $\det(A)$ of a <u>square</u> matrix $A \in M_{nn}$ is defined recursively:

1) If A is 1×1, $\det(A) = [A]_{11}$.

2) If A is $n\times n$, $n>1$, then

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} [A]_{1i} \det(A(1|i))$$

entries of row 1 of A.

$$= [A]_{11}\cdot \det(A(1|1))$$
$$-[A]_{12}\cdot \det(A(1|2))$$
$$+[A]_{13}\cdot \det(A(1|3))$$
$$-[A]_{14} \det(A(1|4))$$
$$+ \cdots$$
$$\pm [A]_{1n} \det(A(1|n)).$$

Example: 1) Let $A = [3]$. $\det(A) = 3$.

2) Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$.

$$\det(A) = [A]_{11} \cdot \det(A(1|1)) - [A]_{12} \det(A(1|2))$$

$$= 3 \cdot \det\left(\underset{\begin{bmatrix} \cancel{3} & \cancel{2} \\ \cancel{1} & 1 \end{bmatrix}}{[1]} \right) - 2 \det\left(\underset{\begin{bmatrix} \cancel{3} & \cancel{2} \\ 1 & \cancel{1} \end{bmatrix}}{[1]} \right)$$

$$= 3 \cdot 1 - 2 \cdot 1$$

$$= 1.$$

3) $$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$$\begin{aligned}
\det(A) &= [A]_{11} \cdot \det(A(1|1)) \\
&\quad - [A]_{12} \det(A(1|2)) \\
&\quad + [A]_{13} \det(A(1|3)) \\
&= 2 \cdot \det\left(\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\right) \\
&\quad - 2 \cdot \det\left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right) \\
&\quad + 1 \cdot \det\left(\begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}\right) \\
&= 2 \cdot \left(2 \cdot \det([2]) - 1 \cdot \det([3])\right)
\end{aligned}$$

$$-2\cdot\left(2\cdot\det([2])-1\cdot\det([0])\right)$$

$$+1\cdot\left(2\cdot\det([3])-2\cdot\det([0])\right)$$

$$=2(2\cdot 2-1\cdot 3)$$

$$-2\cdot(2\cdot 2-1\cdot 0)$$

$$+1(2\cdot 3-2\cdot 0)$$

$$=2-8+6=0.$$

Theorem: Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then $\det(A)=ad-bc$.

Proof: $\det(A)=[A]_{11}\cdot\det(A(1|1))$

$$- [A]_{12} \cdot \det(A(1|2))$$

$$= a \cdot \det([d])$$

$$- b \cdot \det([c])$$

$$= ad - bc.$$

Theorem: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

We skip the proof.

Theorem:

Theorem: It doesn't matter too much which row you use.

Let $A \in M_{n \times n}$. Let $1 \le i \le n$.

$$\det(A) = (-1)^{i+1} [A]_{i1} \det(A(i|1))$$
$$(-1)^{i+2} [A]_{i2} \det(A(i|2))$$
$$(-1)^{i+3} [A]_{i3} \det(A(i|3))$$
$$\cdots$$
$$(-1)^{i+n} [A]_{in} \det(A(i|n)).$$

Example: Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\leftarrow$ use this last row

$$\det(A) = [A]_{31} \det(A(3|1))$$
$$-[A]_{32} \det(A(3|2))$$

$$+[A]_{33}\det(A(3|3))$$

$$\begin{aligned} = &\ 0\cdot\det(\cdots) \\ &-0\cdot\det(\cdots) \\ &+1\cdot\det\left(\begin{bmatrix}3&2\\1&1\end{bmatrix}\right) \end{aligned}$$

$$= 1\cdot(3\cdot 1-2\cdot 1) = 1.$$

Theorem: $\det(A)=\det(A^t)$.

we skip the proof: see Beezer p.267.

Theorem: Let $A\in M_{n\times n}$, $1\le j\le n$.

$$\begin{aligned}\det(A) = &(-1)^{1+j}[A]_{1j}\cdot\det(A(1|j)) \\ &(-1)^{2+j}[A]_{2j}\det(A(2|j)) \\ &(-1)^{3+j}[A]_{3j}\det(A(3|j))\end{aligned}$$

entries

at column j

$$(-1)^{4+j}[A]_{4j}\det(A(4|j))$$

$$\dots$$

$$(-1)^{n+j}[A]_{nj}\det(A(n|j)).$$

Proof: Use $\det(A) = \det(A^t)$, and expand along the j^{th} <u>row</u> of A^t. ($=$ j^{th} column of A).

Example: Let $A = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 3 & 2 & 0 & 1 \end{bmatrix}$

$$\det(A) = [A]_{31} \cdot \det(A(3|1))$$

$$- [A]_{32}\det(A(3|2))$$

$$+ [A]_{33} \det(A(3|3))$$

$$- [A]_{34} \det(A(3|4)) \Big]$$

$$= 0 \cdot \det(\cdots)$$

$$- 1 \cdot \det\left(\begin{bmatrix} 3 & 2 & 1 \\ 3 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}\right)$$

$$+ 0 \cdot \det(\cdots)$$

$$- 0 \cdot \det(\cdots)$$

$$= -1 \cdot \det\left(\begin{bmatrix} 3 & 2 & 1 \\ 3 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}\right)$$

$$= -1 \Big(3(0 \cdot 1 - 1 \cdot 2) - 2 \cdot (3 \cdot 1 - 1 \cdot 3)$$

$$+ 1 \cdot (3 \cdot 2 - 0 \cdot 3) \Big)$$

$$= -1 \big(3(-2) - 2(0) + 1(6) \big)$$

$$= 0.$$

Theorem: Let $A \in M_{n\times n}$. Suppose A has a zero row or zero column. Then $\det(A) = 0$.

Proof: Expand along that row or column:

$$\det(A) = (-1)^{i+1}\cdot 0 \cdot \det(\dots)$$

$$(-1)^{i+2}\cdot 0 \cdot \det(\dots)$$

$$\dots$$

$$(-1)^{i+n}\cdot 0 \cdot \det(\dots)$$

all zero!

$$= 0.$$

Theorem: Suppose $A \in M_{n\times n}$, and A is upper

Triangular.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & & a_{2n} \\ 0 & 0 & a_{33} & \cdots & & a_{3n} \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & a_{n-1,n-1} \\ 0 & \cdots & \cdots & & 0 & a_{nn} \end{bmatrix}$$

Then $\det(A) = a_{11} \cdot a_{22} \cdot a_{33} \cdot \cdots a_{nn}$.

Proof: $\det(A) = [A]_{11} \cdot \det(A(1|1))$

$- [A]_{21} \det(A(2|1))$

$+ \ldots$

$= a_{11} \cdot \det(A(1|1))$

$- 0 \cdot \det(\ldots)$

$$
\begin{aligned}
&+0\cdot\det(\cdots)\\
&-0\cdot\det(\cdots)\\
&\quad\vdots\\
&\pm 0\cdot\det(\cdots)
\end{aligned}
$$

$$
= a_{11}\cdot\det\begin{bmatrix} a_{22} & a_{21} & \cdots & a_{2n} \\ 0 & a_{33} & & \vdots \\ 0 & 0 & a_{44} & \\ \vdots & & & \ddots \\ 0 & \cdots & & a_{nn} \end{bmatrix}.
$$

also upper triangular.

$$
= a_{11}\left(a_{22}\cdot\det\left(\begin{bmatrix} a_{33} & \cdots & & a_{3n} \\ 0 & a_{44} & & \vdots \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}\right)\right)
$$

$$
= a_{11}\cdot a_{22}\cdot a_{33}\cdots a_{nn}
$$

<u>Theorem</u>: Let $A \in M_{n\times n}$. Let B be obtained from A by swapping two columns or rows. Then $\det A = (-1)\cdot\det B$.

We will skip the proof.

<u>Theorem:</u> Let B be obtained from A by multiplying a row of A by $\alpha \in \mathbb{R}$.

Then $\det(B) = \alpha \cdot \det(A)$

Similarly, if B is obtained by mult. a column of A by α, $\det(B) = \alpha \cdot \det(A)$.

Proof: Say B is obtained from A by multiplying row i by $\alpha \in \mathbb{R}$.

Expand along row i:

$\det(B) = (-1)^{i+1} [B] \ \det(B(i|1))$

$$\det(B) = (-1)^{i+1}[B]_{i1}\det(B(i|1))$$
$$(-1)^{i+2}[B]_{i2}\det(B(i|2))$$
$$\vdots$$
$$(-1)^{i+n}[B]_{in}\det(B(i|n))$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= (-1)^{i+1}\alpha[A]_{i1}\det(A(i|1))$$
$$(-1)^{i+2}\alpha[A]_{i2}\det(A(i|2))$$

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{\alpha a_{21} \quad \alpha a_{22} \quad \alpha a_{23}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\cdots$$
$$(-1)^{i+n}\alpha[A]_{in}\det(A(i|n))$$

$$= \alpha(\text{formula for } \det(A))$$
$$= \alpha \cdot \det(A).$$

Theorem: Let $A \in M_{n\times n}$. Suppose B is obtained from A by adding

αR_j to R_i. (one of our standard row operations).

Then $\det(B) = \det(A)$.

Proof: the basic idea of the proof is to expand along row i, and use another theorem about determinants:

<u>Theorem</u>: Let $A \in M_{n \times n}$, with two identical columns or rows. Then $\det(A)=0$.

Example: Let $A = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 2 & 0 \end{bmatrix}$ $\det(A) = 0$.

In summary: each row operation $A \rightsquigarrow B$ changes determinant in a simple way.

- swapping rows: $\det(B) = -\det(A)$.
- multiply row by α: $\det(B) = \alpha \cdot \det(A)$
- add αR_j to R_i : $\det(B) = \det(A)$.

Using these operations, we can turn A into a RREF matrix B.

Example: $A = \begin{bmatrix} 3 & 3 & 2 \\ 3 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

$$A \rightsquigarrow \underset{\boxed{R_2' = R_2 - R_1}}{\begin{bmatrix} 3 & 3 & 2 \\ 0 & -2 & -1 \\ 3 & 2 & 1 \end{bmatrix}} \rightsquigarrow \underset{\boxed{R_3' = R_3 - R_1}}{\begin{bmatrix} 3 & 3 & 2 \\ 0 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix}}$$

$$\rightsquigarrow \underset{\boxed{R_2' = R_2 - 2R_3}}{\begin{bmatrix} 3 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}} \rightsquigarrow \underset{\boxed{R_2 \leftrightarrow R_3}}{\begin{bmatrix} 3 & 3 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} =: B$$

changes det by a sign.

$$\det(B) = 3(-1)(1) = -3.$$

$$\det(A) = -(-3) = 3.$$

Theorem: Suppose $A = \left[\vec{u}_1 \,\middle|\, \cdots \,\middle|\, \vec{x} + \vec{y} \,\middle|\, \cdots \,\middle|\, \vec{u}_n \right]$

i^{th} column.

$$\text{Then } \det A = \det\left(\left[\vec{u}_1 \middle| \cdots \middle| \vec{x} \middle| \cdots \middle| \vec{u}_n\right]\right)$$

$$+ \det\left(\left[\vec{u}_1 \middle| \cdots \middle| \vec{y} \middle| \cdots \middle| \vec{u}_n\right]\right).$$

Proof: expand along the i^{th} column.

Note: In general:

$$\det(A+B) \neq \det(A) + \det(B)$$

Example: Compute: $\det\left(\begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 1 & a_1+b_1 & a_2 & a_3 \\ 1 & a_1 & a_2+b_2 & a_3 \\ 1 & a_1 & a_2 & a_3+b_3 \end{bmatrix}\right)$

$$= \det\left(\begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & b_1 & 0 & 0 \end{bmatrix}\right)$$

$$\begin{vmatrix} 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & b_3 \end{vmatrix}$$

$$= 1 \cdot b_1 \cdot b_2 \cdot b_3 = b_1 b_2 b_3 .$$

A little bit of motivation:

Theorem: Let $A \in M_{n \times n}$. Then A is invertible if and only if $\det(A) \neq 0$.

Proof: A is invertible if and only if

$$A \xrightarrow{RREF} I_n .$$

Each step of the row reduction changes the determinant by 1) $\det(B) = -\det(A)$

2) $\det(B) = \alpha \det(A) \quad \alpha \neq 0.$

3) $\det(B) = \det(A).$

In particular, $\det(B) = 0 \iff \det(A) = 0.$

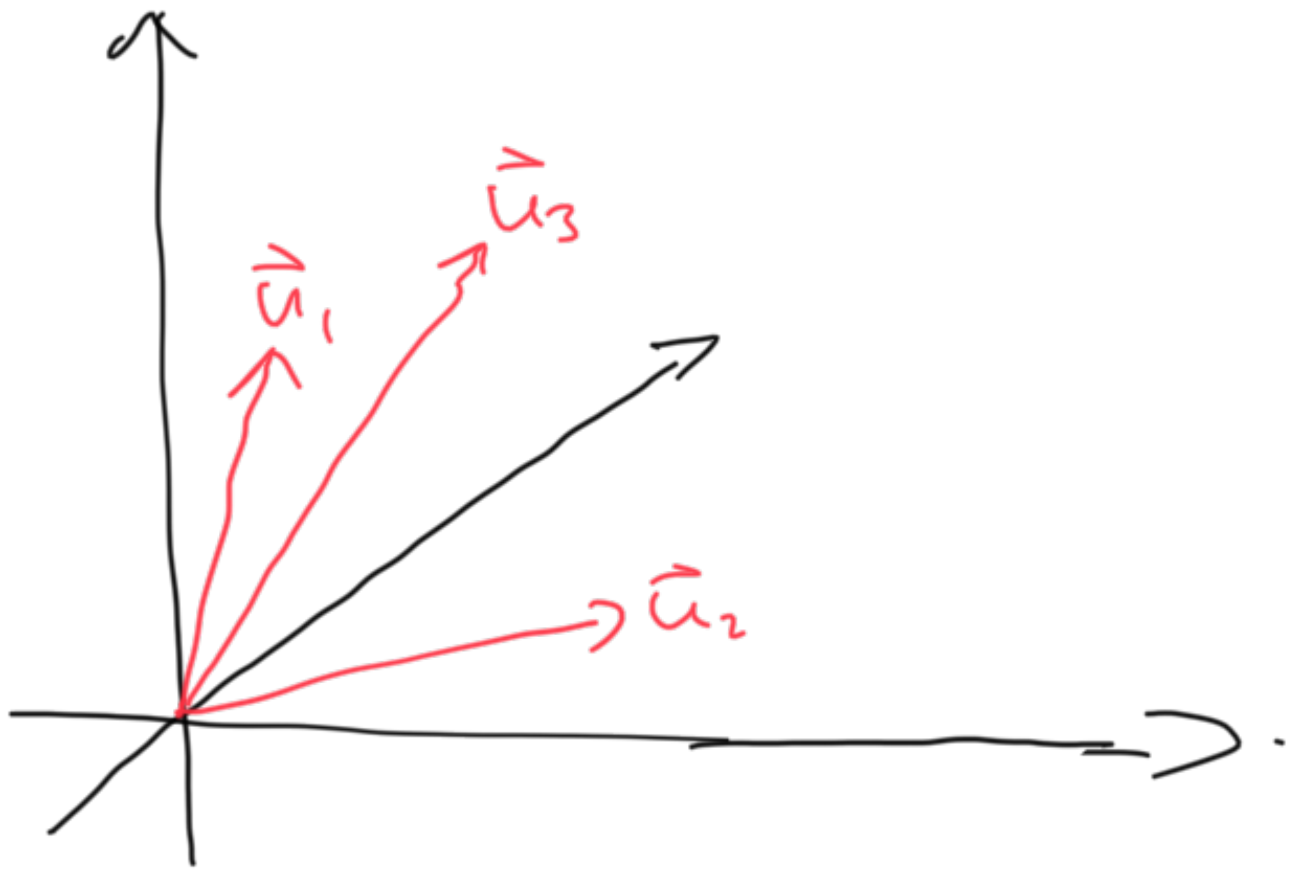
Since $\det(I_n) = \det\left(\begin{bmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \vdots \\ 0 & 0 & 1 & & \\ & \vdots & & 1 & 0 \\ 0 & & \cdots & 0 & 1 \end{bmatrix}\right) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdots 1 = 1,$

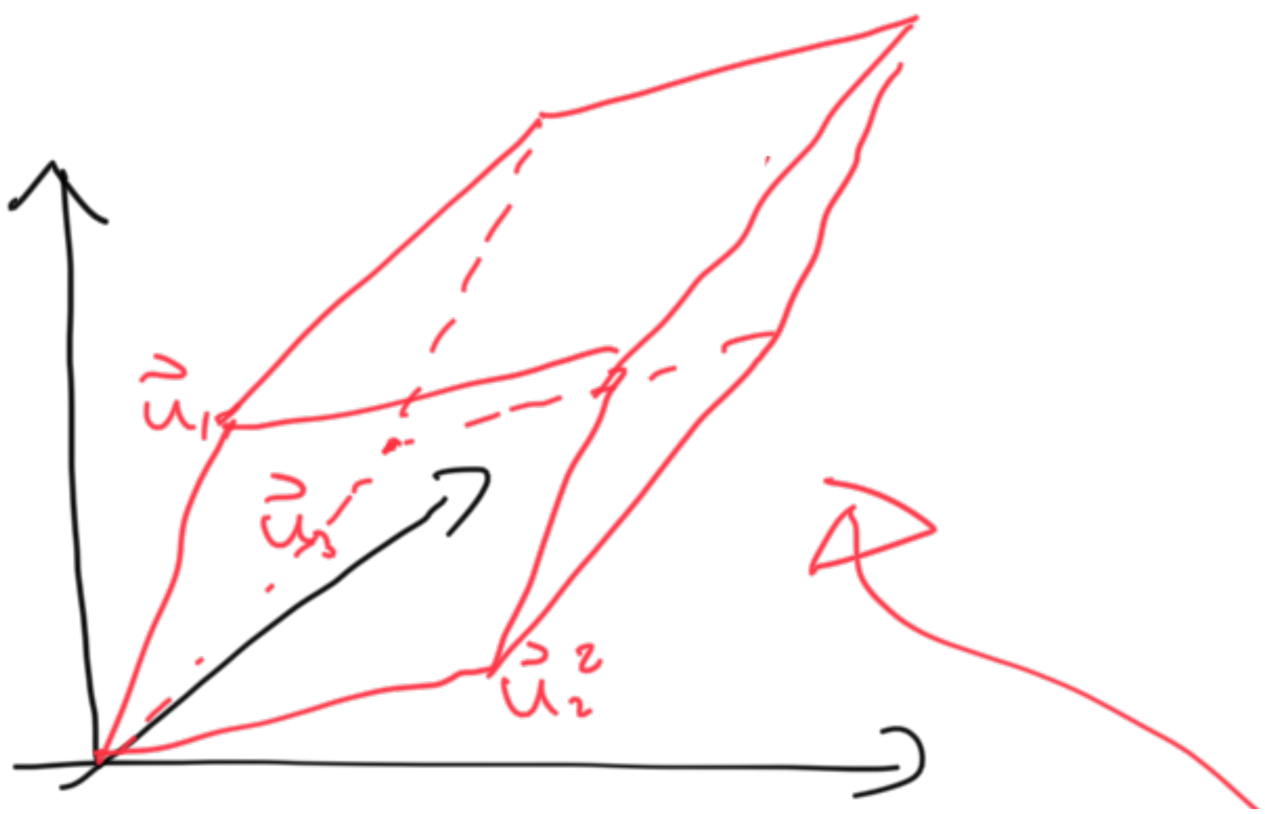
$A \overset{RREF}{\rightsquigarrow} I_n \implies \det(A) \neq 0.$

Suppose, conversely, $\det(A) \neq 0$. Suppose $A \overset{RREF}{\rightsquigarrow} B$ where $B \neq I_n$. Then B must have a zero row. Hence $\det(B) = 0$. But this contradicts our assumption that $\det(A) \neq 0$.

So $A \overset{RREF}{\rightsquigarrow} I_n$. $\square$

Aside: Suppose we have vectors $\vec{u}_1, \vec{u}_2\ \vec{u}_3 \in \mathbb{R}^3$.





$$\det\left(\left[\vec{u}_1 \middle| \vec{u}_2 \middle| \vec{u}_3\right]\right) = \text{volume of}$$

From this perspective, $\det(A) = 0$

$\iff$ the solid built from the columns of A has zero volume.