More about bases

Goal: show that <u>any</u> subspace $V \subset \mathbb{R}^n$ has a basis.

Recall: ① If $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ & $m > n$, then $\vec{v}_1, \dots, \vec{v}_m$ are linearly dependent.

Theorem ②: Let $S = \{\vec{v}_1, \dots, \vec{v}_m\} \in \mathbb{R}^n$.
Suppose S is linearly independent.
Let $\vec{u} \in \mathbb{R}^n$, $\vec{u} \notin \langle S \rangle$.
Then $\langle S, \vec{u} \rangle = \{\vec{v}_1, \dots, \vec{v}_m, \vec{u}\}$ is

also linearly independent.

Proof: Suppose not. (proof by contradiction).

Then $\exists \alpha_1, \dots, \alpha_{m+1}$ s.t.

$$\alpha_1 \vec{v}_1 + \dots + \alpha_m \vec{v}_m + \alpha_{m+1} \vec{u} = \vec{0}_n$$

$$\Rightarrow \alpha_1 \vec{v}_1 + \dots + \alpha_m \vec{v}_m = -\alpha_{m+1} \vec{u}.$$

two possibilities: a) $\alpha_{m+1} = 0$

or

b) $\alpha_{m+1} \neq 0$.

a) $\alpha_1 \vec{v}_1 + \dots + \alpha_m \vec{v}_m = \vec{0}_n$

since $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent,

it follows that $\alpha_1, \dots, \alpha_m = 0$

$$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_m = \alpha_{m+1} = 0$$

b) divided on both sides:

$$-\frac{\alpha_1}{\alpha_{m+1}}\vec{v}_1 + \dots - \frac{\alpha_m}{\alpha_{m+1}}\vec{v}_n = \vec{u}$$

but we <u>assumed</u> $\vec{u} \notin \langle \vec{v}_1, \dots, \vec{v}_m \rangle$.

so we have a contradiction. $\square$

Recall: (3) If $\vec{v}_1, \dots, \vec{v}_n \in V \subset \mathbb{R}^m$, then

$\langle \vec{v}_1, \dots, \vec{v}_n \rangle \subset V$.

Theorem: Let $V \subset \mathbb{R}^n$ be a subspace, $V \neq \{\vec{0}_n\}$.

then V has a <u>basis</u>.

Proof: Pick $\vec{v}_1 \in V$, $\vec{v}_1 \neq 0_{n \times \dots}$. By ③,

$\langle \vec{v}_1 \rangle \subset V$. If $\langle \vec{v}_1 \rangle = V$, then

$\vec{v}$ is a basis of V.

If $\langle \vec{v}_1 \rangle \neq V$, then $\exists \vec{v}_2 \in V$,

$\vec{v}_2 \notin \langle \vec{v}_1 \rangle$.

By ③, $\langle \vec{v}_1, \vec{v}_2 \rangle \subset V$.

By ②, $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

If $\langle \vec{v}_1, \vec{v}_2 \rangle = V$, it is a basis.

If not, $\exists \vec{v}_3 \in V$, $\vec{v}_3 \notin \langle \vec{v}_1, \vec{v}_2 \rangle$.

By ③, $\langle \vec{v}_1, \vec{v}_2, \vec{v}_3 \rangle \subset V$.

By ②, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

If $\langle \vec{v}_1, \vec{v}_2, \vec{v}_3 \rangle = V$, then they are a basis.

Continue in this manner.

Claim: after a finite number of steps, get a basis $\vec{v}_1, \dots, \vec{v}_m$ of V.

Proof of claim: if not, after $n+1$ steps, we would have $n+1$ linearly independent vectors in $\mathbb{R}^n$. By (1), this is impossible. $\square$

Note: In fact, the proof produces a basis of $V \subset \mathbb{R}^n$ with at most n vectors.

Definition: Let $V \subset \mathbb{R}^n$ be a subspace. Let

$\vec{v}_1, \ldots, \vec{v}_m$ be a basis of V.

We call m the "dimension" of V.

Note: this appears to depend on the choice of basis.

It does not, by the following theorem:

Theorem: Let $\vec{u}_1, \ldots, \vec{u}_m$ & $\vec{v}_1, \ldots, \vec{v}_{m'}$ be two basis of $V \subset \mathbb{R}^n$. Then $m = m'$.

Examples: ① dimension of $\mathbb{R}^3$ is 3

(sample basis: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$).

② dimension of $\mathbb{R}^n$ is n.

③ $\dim \mathcal{C}(A) = 2$ $\qquad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

- "Dimension" measures the size of a subspace $V \subset \mathbb{R}^n$.
- If $V \subset \mathbb{R}^n$, dimension of $V \leq n$.

Notation: We write $\dim V$ for the dimension of V.

Definition: Let $A \in M_{mn}$.

We call $\dim \mathcal{C}(A)$ $\quad (\mathcal{C}(A) \subset \mathbb{R}^m)$

the "rank" of A.

big rank
$\mathcal{C}(A)$
big ⌗ of vectors

We call $\dim \mathcal{N}(A)$ $\quad (\mathcal{N}(A) \subset \mathbb{R}^n)$

$\rightsquigarrow$ lots of ...
$\rightsquigarrow$ lots of the form $A\vec{x}$.

the "nullity" of A.

big nullity
$\rightsquigarrow$ big nullspace
$\rightsquigarrow$ lots of $\vec{x}$ s.t. $A\vec{x} = 0_m$.

Example: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Exercise: Show that A has rank $= 1$.
nullity $= 2$.

We'll see, next class, the shortcut to computing these numbers.