Recall the definition of a basis:

Let $W \subset \mathbb{R}^n$ be a subspace.

Let $\vec{u}_1, \dots, \vec{u}_m \in W$.

Then $\vec{u}_1, \dots, \vec{u}_m$ are a basis of W if

1) $W = \langle \vec{u}_1, \dots, \vec{u}_m \rangle$

2) $\vec{u}_1 \dots \vec{u}_m$ are linearly independent.

To check 1) & 2) in examples, we usually:

1) Pick $\vec{w} \in W$, let $A = [\vec{u}_1 | \dots | \vec{u}_m]$.

Need to show that (no matter what $\vec{w}$ we pick)

there is a solution $\vec{x} \in \mathbb{R}^m$ to

$Ax = w$. (can use row reduction).

2) Need to show that $N(A) = \{0_m\}$

Exercise: Let $W = \mathbb{R}^3$. Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Show that $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are a basis of W.

Answer: Need to show: 1) $W = \langle \vec{u}_1, \vec{u}_2, \vec{u}_3 \rangle$

2) $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly independent.

1) Let $A = [\vec{u}_1 | \vec{u}_2 | \vec{u}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let $\vec{w} \in W = \mathbb{R}^3$ $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \end{bmatrix}$

w_3

want to show that there exists a solution $\vec{x} \in \mathbb{R}^3$ to $A\vec{x} = \vec{w}$. Use row-reduction:

goal: show existence of a solution

$$[A|\vec{w}] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & w_1 \\ 0 & 1 & 0 & w_2 \\ 0 & 0 & 1 & w_3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & w_1 - w_2 \\ 0 & 1 & 0 & w_2 \\ 0 & 0 & 1 & w_3 \end{array}\right]$$

so $\vec{x} = \begin{bmatrix} w_1 - w_2 \\ w_2 \\ w_3 \end{bmatrix}$ is a solution. ✓

We conclude that $W = \langle \vec{u}_1, \vec{u}_2, \vec{u}_3 \rangle$.

2) Need to show $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly indep.

$\iff N(A) = \{0_3\}$. Use row reduction:

show

$$[A|0_3] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ \end{array}\right]$$

uniqueness of solution

$$\begin{bmatrix} 0 & 0 & 1 & | & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Hence the only solution to $A\vec{x} = 0_3$ is

$\vec{x} = 0_3$ ✓.

We conclude that $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly independent.

Since 1) & 2) both hold, $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are a basis of W.

Exercise: find two different basis of $W = \mathbb{R}^2$.

$\vec{u}_1$ $\qquad\qquad$ $\vec{v}_1$

$\vec{u}_2$ $\qquad\qquad$ $\vec{v}_2$

basis basis

Basis 1	Basis 2
$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}0\\2\end{bmatrix}$
$\begin{bmatrix}1\\2\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}3\\2\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix}$
$\begin{bmatrix}2\\0\end{bmatrix} \begin{bmatrix}0\\3\end{bmatrix}$	$\begin{bmatrix}5\\4\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$

Exercise: Let $A = \begin{bmatrix}1 & 2 & 3\\ 1 & 1 & 2\\ 1 & 0 & 1\end{bmatrix}$

Find a basis of $\mathcal{N}(A)$.

(Reminder: $A \overset{RREF}{\rightsquigarrow} B$, $\mathcal{N}(A) = \mathcal{N}(B)$, have a nice recipe for producing basis of $\mathcal{N}$).

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\}$$

Room 3

correct nullspace, still need to pick a basis of $\mathcal{N}(A)$

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Room 1

problem: these vectors are not in $\mathcal{N}(A)$

or $\boxed{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$

Room 2

this is a basis of $\mathcal{N}(A)$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Solution: $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix}$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{RREF.}$$

pivotless col. (#3)

Recipe: find one basis vector $\vec{u}_i$ for each pivotless column. $\vec{u}_j = \begin{bmatrix} * \\ * \\ 0 \\ * \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ← pivotless column index.

nonzero only in pivot column indices

← find x_1, x_2

$$\vec{u} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = -1$

$x_2 + x_3 = 0 \qquad x_2 = 1$

$x_1 + x_3 = 0 \qquad x_1 = 1.$

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \checkmark.$$

Example: $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad N(A) \subset \mathbb{R}^4.$

We compute a basis of $N(A)$:

$$A \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & \underline{1} & \underline{-1} \\ 0 & 1 & \underline{0} & \underline{2} \end{bmatrix}$$

pivotless columns 3, 4.

column 3: $\vec{u}_1 = \begin{bmatrix} x_1 \\ x_2 \\ -1 \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow 3 \\ \\ \end{matrix}$ column 4: $\vec{u}_2 = \begin{bmatrix} y_1 \\ y_2 \\ 0 \\ -1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow 4. \end{matrix}$

Solve for $x_1\ x_2$: $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - 1 = 0 \qquad x_1 = 1$

$x_2 = 0$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

Solve for y_1, y_2 $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y_1 + 1 = 0 \quad y_1 = -1$
$y_2 - 2 = 0 \quad y_2 = 2$

$\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \implies \text{basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$

Question: Let $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5 \in \mathbb{R}^2$.
Can they be linearly independant?

Answer: no! $\vec{u}_1, \dots, \vec{u}_5$ are linearly independant

... $A = [\vec{u}_1 | \vec{u}_2 | \cdots | \vec{u}_5]$

if and only if $A = [\vec{u}_1 | \vec{u}_2 | \dots | \vec{u}_5]$

has $N(A) = \{0_5\}$. $\quad A \in M_{2\times 5}$

But: # pivots of $A \leq$ # rows of $A = 2$.

$\to$ $5-2=3$ pivotless columns (>0).

$\to$ there are nonzero vectors in $N(A)$.

So $N(A) \neq \{0_5\}$.

Theorem: Let $\vec{u}_1, \dots, \vec{u}_n \in \mathbb{R}^m$, with $n > m$.

Then $\vec{u}_1 \dots \vec{u}_n$ are linearly dependent.

Proof is by the same argument as in

previous example (show that if

$$A = [\vec{u}_1 | \cdots | \vec{u}_n] \in M_{mn}$$

Then $\mathcal{N}(A) \neq \{0_n\}$, since A has pivotless columns).

Recall: Given $A = [\vec{u}_1 | \cdots | \vec{u}_n] \in M_{mn}$, we call

$\langle \vec{u}_1, \ldots, \vec{u}_n \rangle$ the 'column space' of A.

$\subseteq \mathbb{R}^m$.

Exercise: find a basis of the column space of

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}. \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{u}_3 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Round 1: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Claim: $\vec{u}_1 \notin \langle \vec{v}_1, \vec{v}_2 \rangle$

hence $\langle \vec{v}_1, \vec{v}_2 \rangle \neq \langle \vec{u}_1, \vec{u}_2, \vec{u}_3 \rangle$

Proof: Let $B = \left[\vec{v}_1 \middle| \vec{v}_2 \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$B\vec{x} = \vec{u}_1$$

aug matrix: $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$ ← zero row w/ nonzero constant term $\Rightarrow$ no solution $\vec{x}$.

Second try: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

check consistency

of lin. eq. using row red.

need to check: 0) $\vec{v}_1, \vec{v}_2 \in \langle \vec{u}_1, \vec{u}_2, \vec{u}_3 \rangle$

1) $\langle \vec{v}_1, \vec{v}_2 \rangle = \langle \vec{u}_1, \vec{u}_2, \vec{u}_3 \rangle$???

2) $\vec{v}_1, \vec{v}_2$ are linearly indep.

check uniquess of sol using row reduction.

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0) $\vec{v}_1 \in \langle \vec{u}_1, \vec{u}_2, \vec{u}_3 \rangle$ if there exists $\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{v}_1$.

Solve: $\left[A \,\middle|\, \vec{v}_1 \right] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Big\downarrow$ RREF

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \end{array}\right.$

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 0 & -1 & -1 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

all zero rows have zero cst term $\to$ there exists a solution $\vec{x}$. ✓

$$\Rightarrow \vec{v}_1 \in \langle \vec{u}_1, \vec{u}_2, \vec{u}_3 \rangle.$$

Same with $\vec{v}_2$: $[A|\vec{v}_2] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

as before, all zero rows have zero constant term.

Recall: If B is in RREF, then

$B\vec{x} = \vec{v}$ has a solution

if and only if each zero row of B corresponds to a zero entry of $\vec{v}$.

$$\left[\begin{array}{c|c} B & \vec{v} \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & * & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

To show $\vec{v}_1, \vec{v}_2$ are linearly independent, show if $A = [\vec{v}_1 | \vec{v}_2] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$ then $\mathcal{N}(A) = \{0_2\}$.