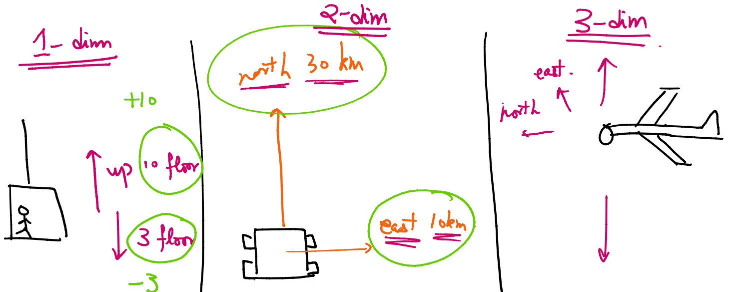
# Chapter 1: Vectors I

Tuesday, September 8, 2020 10:06 PM

What is a vector?

A vector is a math quantities to represent both **magnitude** and **direction**.

e.g. 1-dim | 2-dim | 3-dim




less freedom ← → more freedom

Need one number | Need 2 number | Need 3 numbers

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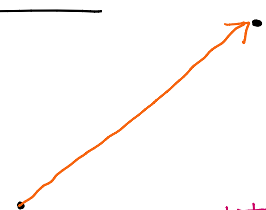
living in $\mathbb{R}^2$:




Given two point A, B we can get a vector

notation for vector ⇝ $\overrightarrow{AB}$

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Notation about vector:

- $\overrightarrow{AB}$
- $\vec{a}$, $\vec{b}$, etc
- $\hat{a}$, $\hat{b}$ : stand for unit vector. (come back to this later)

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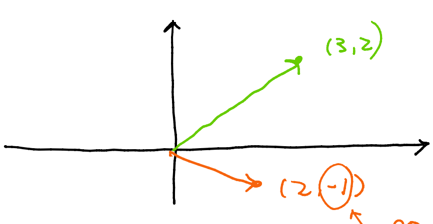
How to calculate a vector? Let say in $\mathbb{R}^2$

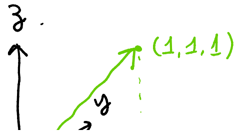
2D)



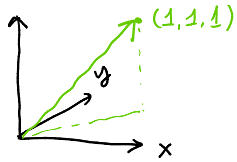

say $\vec{v} = (3, 1)$ means it move 3-units to the right and 2-units to the up.

↖ go down by 1-unit.

3D)



3D

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## § Basic vector operations: 

Work in 2D, i.e. $\mathbb{R}^2$. 

two vectors: $\vec{a} = (a_1, a_2)$ (real numbers)

$\vec{b} = (b_1, b_2)$.

### Operations:

Vector addition: $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2)$.

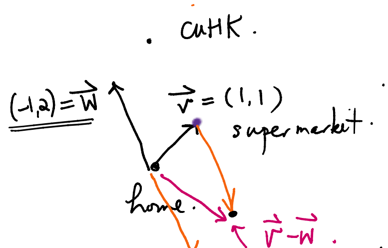
Vector subtraction: $\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2)$.

Scalar multiplication: $k \cdot \vec{a} = (ka_1, ka_2)$.

### Pictorially:

Addition: $\vec{v} = (1,1)$, $\vec{w} = (-1,2)$.

$\vec{v} + \vec{w} = (1-1, 1+2) = (0,3)$.




Subtraction: $\vec{v} - \vec{w} = \vec{v} + (-\vec{w})$

Algebra: $\vec{v} - \vec{w} = (1-(-1), 1-2)$

$= (2, -1)$.

Scalar multiplication: $2\vec{v} = (2 \cdot 1, 2 \cdot 1) = (2,2)$.

$-1 \cdot \vec{v} = (-1, -1)$

$\frac{-1}{2} \cdot \vec{v} = (\frac{-1}{2}, \frac{-1}{2})$

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In particular:
- $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
- $\overrightarrow{AB} = -\overrightarrow{BA}$

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**zero vector:** $\vec{0} = \begin{cases} (0,0) & \text{in } \mathbb{R}^2 \\ (0,0,0) & \text{in } \mathbb{R}^3. \end{cases}$

**Propositions:** $\vec{u}, \vec{v}, \vec{w}$ vectors, and $\alpha, \beta \in \mathbb{R}$.

1. $0 \cdot \vec{v} = \vec{0}$. (0: scalar; $\vec{0}$: zero vector.)
2. $1 \cdot \vec{v} = \vec{v}$.
3. $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ (associative.)
4. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (commutative.)
5. $\vec{0}+\vec{v} = \vec{v}$.
6. $(\alpha+\beta)\vec{v} = \alpha\cdot\vec{v} + \beta\vec{v}$ (distributive).
7. $\alpha(\vec{v}+\vec{w}) = \alpha\cdot\vec{v} + \alpha\cdot\vec{w}$ (distributive).
8. $\alpha\cdot(\beta\vec{v}) = (\alpha\beta)\cdot\vec{v}$.

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**Take rule #6:** $\vec{v} = (v_1, v_2)$.

**say** $\alpha = 2$, $\beta = 3$

$\alpha+\beta = 5$. LHS $= (\alpha+\beta)\cdot\vec{v} = (5v_1, 5v_2)$

**R.H.S.** $= 2\vec{v} + \beta\vec{v} = (2v_1, 2v_2) + (3v_1, 3v_2) = (5v_1, 5v_2)$.

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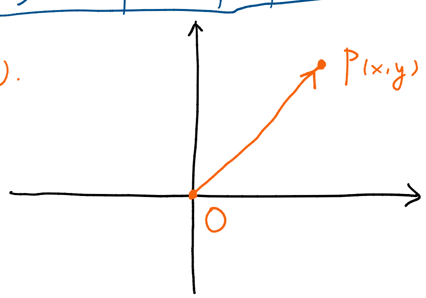
**§ Clarification for position vector:** in $\mathbb{R}^2$.

1).

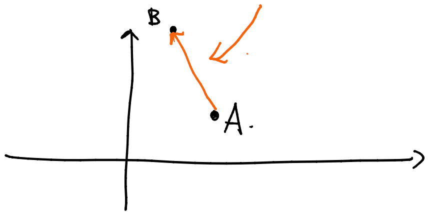



Given a point $P(x,y)$

$\vec{OP} = (x,y)$.

2) $A(a_1, a_2)$, $B(b_1, b_2)$ two points.

$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2)$

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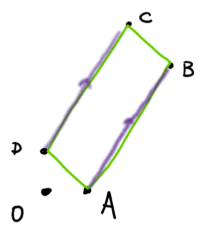
e.g. 2. $A = (1,0)$, $B = (3,3)$, $C = (2,4)$, $D = (0,1)$

Show ABCD is a parallelo-gram.

want $\overrightarrow{AB} = \overrightarrow{DC}$.




$\overrightarrow{AB} = (3-1, 3-0) = (2,3)$

$\overrightarrow{DC} = (2-0, 4-1) = (2,3)$

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## § Length and dot product:

$\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$ in $\mathbb{R}^2$.

norm/Length: $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$ ← always +ve.

Dot product: $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$. (number)

In $\mathbb{R}^3$: $\vec{a} = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$

Length: $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

for vector: $\vec{v}$

one can talk about $\|\vec{v}\|$.

for scalar: $\alpha$

one can talk about $|\alpha|$

In $\mathbb{R}$:

$\vec{v} = (v_1)$

$\|\vec{v}\| = |v_1|$

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## § Length:

Proposition: 1. $\|\vec{v}\| \geq 0$.

2. $\|\vec{v}\| = 0 \iff \vec{v} = \vec{0}$.

explanation: $\vec{v} = (v_1, v_2)$

$\|\vec{v}\|^2 = \left(\sqrt{v_1^2 + v_2^2}\right)^2 = v_1^2 + v_2^2$.

math operation:

$\vec{v} \cdot \vec{w}$ ← eating two vectors produce a scalar.

$\alpha \vec{v}$

$\vec{v} + \vec{w}$.

$\|\vec{v}\|$ ← eating a vector

explanation: $\vec{v} = (v_1, v_2)$

$$\|\vec{v}\|^2 = \left(\sqrt{v_1^2 + v_2^2}\right)^2 = v_1^2 + v_2^2.$$

if $\|\vec{v}\|^2 = 0 \Rightarrow v_1 = 0 = v_2$
$\Rightarrow \vec{v} = \vec{0}.$

$\|\vec{v}\|$ ⟵ eating a vector and produce a scalar.

$|\alpha|$

⋮

3. $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$

4. $\|\alpha\vec{v}\| = |\alpha|\,\|\vec{v}\|$ , $\alpha$ scalar.

($|\alpha|\,\|\vec{v}\|$: multiplication for number.) ($\alpha\vec{v}$: scalar multiplication of vector) ($\|\vec{v}\| = \sqrt{v_1^2+v_2^2}$)

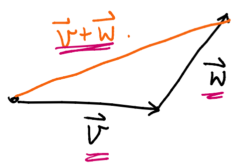
$$\alpha\vec{v} = (\alpha v_1, \alpha v_2)$$

$$\|\alpha\vec{v}\| = \sqrt{(\alpha v_1)^2 + (\alpha v_2)^2} = \sqrt{\alpha^2(v_1^2+v_2^2)} = \sqrt{\alpha^2}\sqrt{(v_1^2+v_2^2)} = |\alpha|\,\|\vec{v}\|$$

($\sqrt{\alpha^2} = |\alpha|$)

✓ 5. $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|.$

explanation in Picture:




**Definition:** • A unit vector is a vector with norm / length 1

notation: $\hat{v}, \hat{w}, \hat{a}, \ldots$

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eg.: $\vec{v} = (1, 2, -2)$

1. find $\|\vec{v}\|$
2. find unit vector in the opposite direction to $\vec{v}$.

Sol: 1. $\|\vec{v}\| = \sqrt{1^2 + 2^2 + (-2)^2}$

$= 3$

2. unit vector in the direction of $\vec{v}$:

$$= \frac{v}{\|\vec{v}\|} = \frac{1}{3}(1, 2, -2) = \left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right).$$

unit vector in the opposite direction of $\vec{v}$:

$$= \frac{-\vec{v}}{\|\vec{v}\|} = \left(\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}\right).$$

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Notation: In $\mathbb{R}^2$, $\hat{i} = (1, 0)$

$\hat{j} = (0, 1)$.

called standard unit vectors.

- You can write $\vec{v} = (v_1, v_2)$

$= v_1\hat{i} + v_2\hat{j}$

In $\mathbb{R}^3$: $\hat{i} = (1, 0, 0)$

$\hat{j} = (0, 1, 0)$

$\hat{k} = (0, 0, 1)$.

$$\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k} = (v_1, v_2, v_3).$$

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## § Dot product:

$\vec{u}, \vec{v}, \vec{w}$ vectors

$\alpha, \beta \in \mathbb{R}$.

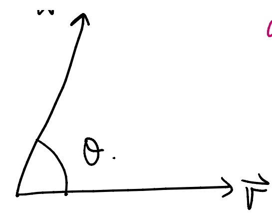
Proposition:

check by yourself

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $(\alpha\vec{u} + \beta\vec{v}) \cdot \vec{w} = \alpha(\vec{u} \cdot \vec{w}) + \beta(\vec{v} \cdot \vec{w})$.
3. $\vec{0} \cdot \vec{v} = 0$.
4. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.
5. $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$.

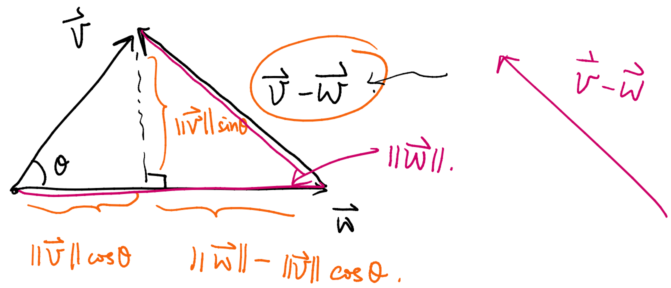
$\vec{w}$

angle between $\vec{v}$ and $\vec{w}$.




angle between $\vec{v}$ and $\vec{w}$.

## Pf of 5. :




$$\|\vec{v}-\vec{w}\|^2 = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$$

$$= \vec{v}\cdot(\vec{v}-\vec{w}) - \vec{w}\cdot(\vec{v}-\vec{w})$$

$$= \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{w} - \vec{w}\cdot\vec{v} + \vec{w}\cdot\vec{w}$$

$$= \|\vec{v}\|^2 - 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2.$$

$$\|\vec{v}-\vec{w}\|^2 = \left(\|\vec{w}\| - \|\vec{v}\|\cos\theta\right)^2 + \left(\|\vec{v}\|\sin\theta\right)^2.$$

$$= \|\vec{w}\|^2 - 2\|\vec{w}\|\|\vec{v}\|\cos\theta + \|\vec{v}\|^2\cos^2\theta + \|\vec{v}\|^2\sin^2\theta$$

$$\|\vec{v}\|^2(\cos^2\theta + \sin^2\theta) \quad 1. \quad (\cos^2\theta + \sin^2\theta = 1.)$$

$$= \|\vec{w}\|^2 - 2\|\vec{w}\|\|\vec{v}\|\cos\theta + \|\vec{v}\|^2.$$

$$\Longrightarrow \quad \|\vec{w}\|\|\vec{v}\|\cos\theta = \vec{v}\cdot\vec{w}$$

if $\vec{w}, \vec{v}$ are unit vector

then $\vec{w}\cdot\vec{v} = \cos\theta$.