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Soft Network Coding in Wireless Two-Way Relay Channels

Shengli Zhang[*+], Yu Zhu[†], Soung-chang Liew[*]

[*]Department of Information Engineering
The Chinese University of Hong Kong
New Territories, Hong Kong

[†]Department of Communication Science and Engineering
Fudan University, Shanghai, China

[+]Department of Communication Engineering
Shenzhen University, Shenzhen, China

Abstract

Application of network coding in wireless Two-Way Relay Channels (TWRC) has received much attention recently because its ability to improve throughput significantly. In traditional designs, network coding operates at upper layers above (including) the link layer and it requires the input packets to be correctly decoded. However, this requirement may limit the performance and application of network coding due to the unavoidable fading and noise in wireless networks. In this paper, we propose a new wireless network coding scheme for TWRC, which is referred to as soft network coding (SoftNC), where the relay nodes applies symbol-by-symbol soft decisions on the received signals from the two end nodes to come up with the network coded information to be forwarded. We do not assume further channel coding on top of SoftNC at the relay node (channel coding is assumed at the end nodes). According to measures of the soft information adopted, two kinds of SoftNC are proposed: amplify-and-forward SoftNC (AF-SoftNC) and soft-bit-forward SoftNC (SBF-SoftNC). We analyze the both the ergodic capacity and the outage capacity of the two SoftNC schemes. Specifically, analytical form approximations of the ergodic capacity and the outage capacity of the two schemes are given and validated. Numerical simulation shows that our SoftNC schemes can outperform the traditional network coding based two-way relay protocol, where channel decoding and re-encoding are used at the relay node. Notable is the fact that performance improvement is achieved using only simple symbol-level operations at the relay node.

***Keywords*:** Network coding, channel coding, wireless two-way relay channel, log-likelihood ratio, soft bit, ergodic capacity, outage capacity.

I. INTRODUCTION

Wireless network coding has recently received much attention from the research community because of the advantages which it inherits from the network coding design in wired networks. These advantages include improved network throughput, robustness, and security. The broadcast nature of the wireless medium allows network coding to be applied in an even more natural way, since one transmission of the network-coded information can reach several target receivers simultaneously. On the short side, however, is the fact that wireless networks are affected by several attributes more much severely than in wired networks, such as multi-path fading, power constraint, limited channel bandwidth, and changing topology. How to make best use of the wireless medium while dealing with its shortcomings has been an active research issue for a long time. The advent of network coding brings a new angle to the attack of this problem.

Two-way relay channel (TWRC) is a typical and important basic wireless network topology. An obvious approach to TWRC is to use the same relay protocols as in the one-way relay channel (OWRC) [1, 2]. The application of network coding in TWRC was first proposed in [4], where it was shown that by performing the network coding combination in the third time slot at the relay node, the whole network needs only three time slots (saves one time slot when compared to the traditional relay protocols in [1, 2]) to complete the information exchanging between the two end nodes. Such relay protocol has been implemented in [5] with software defined radio and it is shown that substantial performance improvement can be achieved. Ref. [6] and [7] studied the joint network coding and channel coding designs in TWRC, with the effects of wireless channel fading taken into account.

Previous designs of network coding for TWRC [4] require correct channel decoding of the received packets from the two ends at the relay node, which may limit the throughput of TWRC. This is similar to that in OWRC, where the performance of the decode-and-forward protocol may be much worse than the performance of the amplify-and-forward protocol under certain scenarios [8]. Furthermore, due to the time variations of the channel fading, it cannot be always assumed that the received packet is decoded correctly, especially when the channel is in deep fading. In addition, in some situations, power consumption at the relay node is a concern (e.g., the relay node is a normal user with limited battery power) and the channel decoding processing may consume excessive amount of power.

In this paper, to remove the requirement of channel decoding, we propose a new wireless network coding scheme, referred to as soft network coding (SoftNC), where the relay node applies symbol-by-symbol soft decisions on the received signals from the two end nodes to come up with the network-coded information to be forwarded. Note that channel coding is only performed at the end nodes but not the relay node. In particular, the relay node does not perform channel decoding and re-encoding, and that channel coding is on an end-to-end basis where only the end nodes are involved in channel coding and decoding. In SoftNC, the forwarded signal is actually the soft information of the bits that are obtained by doing the XOR operation to the two codewords received respectively from the two end nodes. According to measures of soft information adopted, two kinds of SoftNC are proposed: amplify-and-forward SoftNC (AF-SoftNC) and soft-bit-forward SoftNC (SBF-SoftNC). In the former, the log-likelihood ratios (LLR) of the bits are generated and forwarded; in the latter, the soft bits (i.e., the MMSE estimation of the XOR-ed bit) are generated and forwarded.

This paper also analyzes the performance of the two proposed SoftNC schemes in terms of ergodic capacity and outage capacity, which are defined for fast fading channel and slow fading channel, respectively. We provide analytical form approximations of the ergodic capacity and the outage capacity of the two SoftNC schemes. It is shown that the analytical results are very close to the true simulated information rates that are obtained according to the definition of mutual information. Our simulation shows that, AF-SoftNC and SBF-SoftNC can obtain substantial performance improvements over the conventional two-way relay protocols with or without network coding. Since the proposed SoftNC design also does not require any channel decoding and re-encoding processing at the relay node, it is a very promising network coding method in terms of actual practice in wireless networks.

Related Work:

The fundamental idea behind the proposed SoftNC design is that due to the unreliability of the wireless fading channels instead of forwarding the decoded-and-network-coded (XOR-ed) bit, the relay node can calculate-and-forward the likelihood information, i.e., how likely the network coded bit is “0” or “1”. The proposed AF-SoftNC design has been considered in a preliminary version of this paper [7]. The same similar idea was independently proposed in a two sources relay system in [9]. More recently, an encoding-decoding framework and BER analysis in fading channel for the two-source relay system have been considered in [10]. Different from these works, where the soft information is obtained based on the whole received packet (for example, after the soft-input soft-output channel decoding), our work focuses on the network coding where the relay directly obtains the symbol-by-symbol soft information of the network coded bit based on the received signals from the two end nodes without any channel coding operation. This greatly reduces the computational complexity at the relay node since the channel decoding processing occupies most of the baseband power.

Physical layer Network Coding (PNC) [11], including forwarding the likelihood information of the XOR-ed information from the received superimposed signal [12, 13], completes the network coding operation in only two time slots in TWRC. In [14, 15], it was proved that PNC can approach the capacity of TWRC with Gaussian noise in low SNR and high SNR region respectively. In [16], T. Wang and G. B. Giannakis proposed another PNC based scheme which requires two time slots, where the relay estimates individual source symbols from the superimposed signal and combines them over complex field rather than Galois field before forwarding. However, the implementation of these PNC-based schemes is complex.

The rest of this paper is organized as follows. Section II presents the system model. In Section III, we present two soft network coding designs, AF-SoftNC and SBF-SoftNC. We analyze their capacity in Section IV. Section V presents our numerical simulation results. Section VI concludes this paper.

II. SYSTEM MODEL

Consider a two-way relay communication system as shown in Fig. 1, where the two end nodes, N_1 and N_2, exchange their information with the help of the relay node N_3. We also assume that different transmissions among the three nodes are separated in non-overlapping time slots. Since it is difficult to implement a full duplex wireless transceiver due to the large difference between the transmitted signal power and received signal power [2], we assume that all the three nodes work in the half-duplex mode, where each node either transmits or receives at a particular time. Due to the broadcast nature of the wireless medium, packets transmitted by any node can be received by the other two nodes. In the first slot, node N_1 sends its packet to both node N_2 and the relay node N_3. In the second time slot, node N_2 sends its packet to both node N_1 and node N_3. If network coding is used at the relay node N_3, in the third time slot node N_3 will combine the two packets received in the previous two time slots with network coding and forward the network coded packet to the other two nodes. If network coding is not used, node N_3 will forward the two received packets in the third and forth time slots, respectively.

Let $U_i = \left[u_i[0],\cdots,\ u_i[n],\cdots,\ u_i[K_i-1]\right]$ [1] denote the information packet transmitted by the two end nodes N_i, where i=1,2, $u_i[n]\in\{0,1\}$, and K_i is the corresponding packet length. Channel coding (including interleaving) is usually performed for certain transmission reliability in wireless channels. Let Γ_i denote the channel coding scheme at node N_i, and let $D_i = \left[d_i[0],\cdots,\ d_i[n],\cdots,\ d_i[M_i-1]\right]$ denote the codeword, where $d_i[n]\in\{0,1\}$ and M_i is the codeword length. For simplicity, we assume that the two end nodes use the same coding schemes, i.e., $\Gamma \equiv \Gamma_1 = \Gamma_2$, and the same packet length i.e., $K \equiv K_1 = K_2$, and $M \equiv M_1 = M_2$ [2]. We further assume that BPSK modulation is used, and then the relationship between a BPSK symbol and the corresponding coded bit is given by

$$x_i[n] = 1 - 2d_i[n]. \tag{1}$$

In the following, we define that Γ_i includes both channel coding and BPSK modulation. The relationship between the information packet and the transmitted BPSK packet can be represented by

[1] Throughout this paper, we use upper case letters to denote packets and the corresponding lower case letters to denote the symbols in the packets.
[2] We will discuss the system design with different channel coding schemes at the two end nodes in Part C Section III.

$$X_i = \Gamma_i\left(U_i\right) \qquad U_i = \Gamma_i^{-1}\left(X_i\right) \tag{2}$$

where Γ_i^{-1} denotes the decoding processing. We could consider the whole codeword as being divided into $L \geq 1$ blocks with the block length Q less than or equal to the length of the channel coherence time (i.e., at least Q symbols are covered during the coherence time). The received signal in the l-th block at node N_j can be expressed as

$$y_{i,j}^l[m] = h_{i,j}^l x_i^l[m] + w_{i,j}^l[m] \qquad \text{for } i,\ j = 1,2,3 \ \text{ and } m = 0,\ldots,Q-1 \tag{3}$$

where $x_i^l[m]$ is the mth symbol in the lth block transmitted by node N_i; $y_{i,j}^l[m]$ is the corresponding signal received at node N_j; $w_{i,j}^l[m]$ is the corresponding complex Gaussian noise at node N_j with unit variance per dimension, i.e., $w_{i,j}^l \sim \mathcal{CN}\left(0,1\right)$; and $h_{i,j}^l$ is the corresponding channel fading coefficient. It should be noted here that throughout this paper, the transmit power is normalized to one and $h_{i,j}^l$ actually includes the real transmit power, the path loss effect, and the small-scale multipath fading effect. When the block number is large and a random interleave is applied among the blocks, we can assume that $h_{i,j}^l$ is independent identical complex Gaussian distributed for different blocks with the variance $\lambda_{i,j} = E\{|h_{i,j}|^2\}$.

Next, we briefly review the traditional straightforward network coding (SNC) scheme [4], where the network coding operation is performed at the relay node N_3, as shown in Fig. 2. Assume that node N_3 has perfect channel state information (CSI) of $h_{i,3}^l$. By performing coherent demodulation to the received signal from the end node N_i, we have

$$\tilde{y}_{i,3}^l[m] \triangleq \mathrm{Re}\left\{\frac{\left(h_{i,3}^l\right)^*}{\left|h_{i,3}^l\right|} y_{i,3}^l[m]\right\} = \left|h_{i,3}^l\right| x_i^l[m] + \tilde{w}_{i,3}^l[m] \tag{4}$$

where with the distribution $\tilde{w}_{i,3}^l[m] \sim \mathcal{N}\left(0,\ 1\right)$. Since soft decoding is considered in the system, two kinds of soft information can be generated as the measure of the detection result. These are the Log-Likelihood Ratio (LLR) value [6, 17, 18]

$$v_{i,3}^l[m] = \ln\left(\frac{P\left(x_i^l[m]=1 \mid \tilde{y}_{i,3}^l[m]\right)}{P\left(x_i^l[m]=-1 \mid \tilde{y}_{i,3}^l[m]\right)}\right) = \ln\left(\frac{P\left(\tilde{y}_{i,3}^l[m] \mid x_i^l[m]=1\right)}{P\left(\tilde{y}_{i,3}^l[m] \mid x_i^l[m]=-1\right)}\right) = 2\left|h_{i,3}^l\right| \tilde{y}_{i,3}^l[m] \tag{5}$$

and the soft bit value [19]

$$v_{i,3}^{l}[m]=P\left(x_i^l[m]=1\,|\,\tilde{y}_{i,3}^{l}[m]\right)-P\left(x_i^l[m]=-1\,|\,\tilde{y}_{i,3}^{l}[m]\right)$$
$$=\frac{\exp\left(2\left|h_{i,3}^{l}\right|\tilde{y}_{i,3}^{l}[m]\right)-1}{\exp\left(2\left|h_{i,3}^{l}\right|\tilde{y}_{i,3}^{l}[m]\right)+1}=\tanh\left(\left|h_{i,3}^{l}\right|\tilde{y}_{i,3}^{l}[m]\right)\quad. \tag{6}$$

By sending the soft information to the channel decoder, the decoded packets are given by

$$\hat{U}_1=\Gamma^{-1}\left(V_{1,3}\right)\qquad \hat{U}_2=\Gamma^{-1}\left(V_{2,3}\right). \tag{7}$$

If both packets are decoded correctly, node N_3 performs the network coding operation by combining the two information packets as follows

$$U_3=\hat{U}_1\oplus\hat{U}_2 \tag{8}$$

where "$\oplus$" denotes the XOR operation[3]. Finally, the network coded packet U_3 is channel encoded (also by Γ), modulated, and forwarded to both node N_1 and N_2, as shown in Fig. 2.

During the three time slots, every end node receives two packets. One packet is received from its counterpart node in either the first or second time slot and the other packet is from the relay node in the third time slot. The channel-decoding processing of the two packets is the same for the two end nodes. Take node N_2 as an example. It receives $Y_{1,2}$ from node N_1 in the first time slot and $Y_{3,2}$ from node N_3 in the third time slot. After removing the self-information in $Y_{3,2}$, N_2 obtains a noise corrupted codeword of U_1, which is referred to as $\hat{Y}_{3,2}$. It can be seen that actually $Y_{1,2}$ and $\hat{Y}_{3,2}$ are the two received independent copies of the codeword X_1. N_2 can perform the maximum ratio combination (MRC) to $Y_{1,2}$ and $\hat{Y}_{3,2}$ before channel decoding.

[3] It is shown in [3] that the general network coding combination is the linear operation over a finite Field. The addition over GF(2), i.e., XOR operation, is usually considered in practical networks for its simplicity and good performance [5].

III. SOFT NETWORK CODING DESIGN

In this section, we first introduce the basic idea of SoftNC, and then propose the SoftNC design for practical systems when fading and noise effects are considered. Finally, we discuss the SoftNC design in TWRC when the two end nodes use different channel coding schemes.

A. Basic idea

As shown in Section II, in the SNC scheme, the network combination is performed after the successful decoding of the packets from the two end nodes. However, due to the wireless channel fading effect, the received packet may not always be decoded successfully. Furthermore, in some situation, for example, when the relay node is a normal mobile user, the battery power is limited and should be used as efficient as possible. However, the channel decoding processing is power hungry, especially when advanced channel codes, such as Turbo codes and LDPC codes, are used.

The basic idea of the proposed SoftNC scheme is founded on the linear property of the channel code. That is, the linear combination of the two codewords, which are generated from exactly the same coding scheme with the same length, is actually another codeword. This linearity can be formulated as[4]

$$\Gamma\left(U_1 \oplus U_2\right) = \Gamma\left(U_1\right) \odot \Gamma\left(U_2\right). \tag{9}$$

Almost all practical wireless channel codes, such as convolutional codes, Turbo codes, and LDPC codes, are linear codes. By applying this property of the channel codes, and the fact that network coding is also a linear mapping, it is easily seen that the network coding combination can be done on the codewords. This motivates the proposed SoftNC scheme, as shown in Fig. 3. By carefully combining the soft decisions $V_{1,3}$ and $V_{2,3}$, the output packet of SoftNC, denoted by V_3, is in fact the codeword of the target information packet $U_1 \oplus U_2$. For the simplicity of explanation, if we ignore the noise and fading effects, the SoftNC design can be expressed as

$$U_3 = \hat{U}_1 \oplus \hat{U}_2 = U_1 \oplus U_2 = \Gamma^{-1}\left(X_1\right) \oplus \Gamma^{-1}\left(X_2\right) \tag{10}$$

$$X_3 = \Gamma\left(U_3\right) = \Gamma\left(\Gamma^{-1}\left(X_1\right) \oplus \Gamma^{-1}\left(X_2\right)\right) = X_1 \odot X_2. \tag{11}$$

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[4] In this equation, the notation "$\odot$" represents the element-wise product. As shown in (1), the channel coding mapping function Γ is from the information packet to the transmitted packet. The logical "XOR" operation on two binary bits in the codeword *D* is equivalent to the multiplication operation on two BPSK symbols in the transmitted packet *X*.

By comparing SNC in Fig. 2 with SoftNC in Fig. 3, we see that the relay in SoftNC performs network coding without any channel decoding process, while the SNC scheme requires two channel decoding processes and one channel encoding process. Since most of the power in baseband signal processing is consumed by the channel decoding, SoftNC can greatly increase the power efficiency of relay nodes in wireless networks.

B. SoftNC design in fading channels

In practical wireless channels, the data transmission is unavoidably affected by noise and fading. Before considering these effects in SoftNC, we first review the memoryless relay protocols in the well-known OWRC [20]. That is, we just consider the relay direction $N_1 \to N_3 \to N_2$ in Fig. 1. According to (4), the received packet at N_3 after coherent demodulation is given by

$$\tilde{y}_{1,3}^l[m] = \left|h_{1,3}^l\right| x_1^l[m] + \tilde{w}_{1,3}^l[m]\,. \tag{12}$$

In the literature of OWRC, there are mainly two kinds of memoryless relay protocols [20], where the signal to be forwarded can be expressed as an estimate of $x_1^l[m]$ from the received signal $\tilde{y}_{1,3}^l[m]$. The estimated signals in the two relay protocols correspond to the two soft information measures in (5) and (6), respectively. Specifically,

1) Amplify and forward (AF)

$$v_{1,3}^l[m] = \ln\left(\frac{P\left(x_1^l[m]=1\left|\tilde{y}_{1,3}^l[m]\right.\right)}{P\left(x_1^l[m]=-1\left|\tilde{y}_{1,3}^l[m]\right.\right)}\right) = 2\left|h_{1,3}^l\right|\tilde{y}_{1,3}^l[m]\,. \tag{13}$$

It can be seen from (13) that in the AF protocol the estimated signal is actually the LLR [21] of the transmitted signal from N_1.

2) Soft bit forward (SBF)

$$v_{1,3}^l[m] = P\left(x_1^l[m]=1\left|\tilde{y}_{1,3}^l[m]\right.\right) - P\left(x_1^l[m]=-1\left|\tilde{y}_{1,3}^l[m]\right.\right) = \tanh\left(\left|h_{1,3}^l\right|\tilde{y}_{1,3}^l[m]\right). \tag{14}$$

The SBF protocol was referred to as estimate-and-forward protocol in [20]. It can be seen that SBF in (14) is actually the conditional expectation of the hard bit $E\left(x_1^l[m]\left|\tilde{y}_{1,3}^l[m]\right.\right)$, i.e. the MMSE estimation, at the relay node. It was proved that SBF is optimal in terms of maximizing GSNR (general SNR [20]) of the received signal at the sink and maximizing the mutual information between the source and the sink in the conventional OWRC [22].

Finally, in each protocol, the estimated signal is normalized

$$x_3^l[m] = \sqrt{\frac{1}{E\left\{\left(v_{1,3}^l[m]\right)^2\right\}}}\, v_{1,3}^l[m] \tag{15}$$

and forwarded to the destination node N_2.

In SoftNC, the two estimation methods above are extended to TWRC. The difference now is that the relay node estimates $\left(x_1^l[m]\cdot x_2^l[m]\right)$[5] (rather than individual $x_i^l[m]$) with the knowledge of $\tilde{y}_{1,3}^l[m]$ and $\tilde{y}_{2,3}^l[m]$.

1) AF-SoftNC

As shown in (13), AF in OWRC actually can be seen as forwarding the LLR of the received bits to the destination node. In TWRC, the forwarded signal, i.e., the LLR of the network coded bit $\left(x_1^l[m]\cdot x_2^l[m]\right)$, is given by

$$\begin{aligned} v_3^l[m] &= \ln\frac{P\left(x_1^l[m]\cdot x_2^l[m]=1 \middle| \tilde{y}_{1,3}^l[m], \tilde{y}_{2,3}^l[m]\right)}{P\left(x_1^l[m]\cdot x_2^l[m]=-1 \middle| \tilde{y}_{1,3}^l[m], \tilde{y}_{2,3}^l[m]\right)} \\ &= \ln\frac{P\left(x_1^l[m]=1, x_2^l[m]=1 \middle| \tilde{y}_{1,3}^l[m], \tilde{y}_{2,3}^l[m]\right) + P\left(x_1^l[m]=-1, x_2^l[m]=-1 \middle| \tilde{y}_{1,3}^l[m], \tilde{y}_{2,3}^l[m]\right)}{P\left(x_1^l[m]=1, x_2^l[m]=-1 \middle| \tilde{y}_{1,3}^l[m], \tilde{y}_{2,3}^l[m]\right) + P\left(x_1^l[m]=-1, x_2^l[m]=1 \middle| \tilde{y}_{1,3}^l[m], \tilde{y}_{2,3}^l[m]\right)}. \\ &= \ln\left(\frac{1+\exp\left(2\,|\,h_{1,3}^l\,|\,\tilde{y}_{1,3}^l[m] + 2\,|\,h_{2,3}^l\,|\,\tilde{y}_{2,3}^l[m]\right)}{\exp\left(2\,|\,h_{1,3}^l\,|\,\tilde{y}_{1,3}^l[m]\right)+\exp\left(2\,|\,h_{2,3}^l\,|\,\tilde{y}_{2,3}^l[m]\right)}\right) \end{aligned} \tag{16}$$

It can be found that (16) is equivalent to "Boxplus" operation in the channel decoding algorithm as in [21]. As widely used in channel decoding algorithms [23], (16) can be further approximated as

$$v_3^l[m] \approx \operatorname{sign}\left(\tilde{y}_{1,3}^l[m]\cdot\tilde{y}_{2,3}^l[m]\right)\min\left\{2\,|\,h_{1,3}^l\,|\,\tilde{y}_{1,3}^l[m], 2\,|\,h_{2,3}^l\,|\,\tilde{y}_{2,3}^l[m]\right\}. \tag{17}$$

The mathematics behind this approximation is complicated. An insight of this approximation is that since $v_3^l[m]$ is an estimate of $x_1^l[m]x_2^l[m]$ from $\tilde{y}_{1,3}^l[m]$ and $\tilde{y}_{2,3}^l[m]$, its signature must be $\operatorname{sign}\left(\tilde{y}_{1,3}^l[m]\cdot\tilde{y}_{2,3}^l[m]\right)$ and its belief must not be larger than either $\tilde{y}_{1,3}^l[m]$ or $\tilde{y}_{2,3}^l[m]$.

2) Soft bit forward (SBF)

[5] Note that the relay can also estimate the complex field network coding (CFNC) form of $x_1^l[m]$ and $x_2^l[m]$ as in [16]. However, in our TWRC model where every end node knows its own information, the CFNC's advantage of being able to decode both individual symbols from the CFNC symbol is not needed: in fact, decoding the individual symbols explicitly results in a power penalty in CFNC. Therefore, we only consider finite field network coding in our paper.

Based on the definition of soft bit in (6) and (14), in the proposed SBF-SoftNC design in TWRC, the soft bit of $x_1^l[m]\cdot x_2^l[m]$ is defined by

$$v_3^l[m]=P\left(x_1^l[m]\cdot x_2^l[m]=1\middle|\tilde{y}_{1,3}^l[m],\tilde{y}_{2,3}^l[m]\right)-P\left(x_1^l[m]\cdot x_2^l[m]=-1\middle|\tilde{y}_{1,3}^l[m],\tilde{y}_{2,3}^l[m]\right). \tag{18}$$

It can be proved that

$$\begin{aligned} v_3^l[m]&=\frac{\dfrac{P\left(x_1^l[m]\cdot x_1^l[m]=1\middle|\tilde{y}_{1,3}^l[m],\tilde{y}_{2,3}^l[m]\right)}{P\left(x_1^l[m]\cdot x_1^l[m]=-1\middle|\tilde{y}_{1,3}^l[m],\tilde{y}_{2,3}^l[m]\right)}-1}{\dfrac{P\left(x_1^l[m]\cdot x_1^l[m]=1\middle|\tilde{y}_{1,3}^l[m],\tilde{y}_{2,3}^l[m]\right)}{P\left(x_1^l[m]\cdot x_1^l[m]=-1\middle|\tilde{y}_{1,3}^l[m],\tilde{y}_{2,3}^l[m]\right)}+1}\\ &=\frac{\exp\left(2\,|\,h_{1,3}^l\,|\,\tilde{y}_{1,3}^l[m]\right)-1}{\exp\left(2\,|\,h_{1,3}^l\,|\,\tilde{y}_{1,3}^l[m]\right)+1}\bullet\frac{\exp\left(2\,|\,h_{2,3}^l\,|\,\tilde{y}_{2,3}^l[m]\right)-1}{\exp\left(2\,|\,h_{2,3}^l\,|\,\tilde{y}_{2,3}^l[m]\right)+1}.\\ &=\tanh(|\,h_{1,3}^l\,|\,\tilde{y}_{1,3}^l[m])\tanh(|\,h_{2,3}^l\,|\,\tilde{y}_{2,3}^l[m]) \end{aligned} \tag{19}$$

It can be seen from (19) that the soft bit of $x_1^l[m]\cdot x_2^l[m]$ is in fact the product of the soft bit of $x_1^l[m]$ and that of $x_2^l[m]$.

Similar to the one-way relay case, the estimate of the network coded bit $\left(x_1^l[m]\cdot x_2^l[m]\right)$ in (16) and (19) is normalized according to the power constraint at the relay node

$$x_3^l[m]=\alpha v_3^l[m]=\sqrt{\frac{1}{E\left\{\left(v_3^l[m]\right)^2\right\}}}v_3^l[m]. \tag{20}$$

Finally, the output of SoftNC, $x_3^l[m]$, is broadcast to both end nodes.

C. Generalization

The above discussion on SoftNC is based on the assumption that the two end nodes have the same channel coding scheme. In this subsection we show that the proposed SoftNC design is still applicable when the two end nodes have different channel coding schemes.

Let Γ_i, K_i, and M_i denote the channel coding scheme, the information packet length and the transmitted packet length, respectively, at node N_i, for i=1,2. We assume that they are different for different nodes except that M_1=M_2=M. Note that when the codeword lengths are different at the two end nodes, a group of zero symbols can be inserted at the end of the shorter coded packet. After performing the SoftNC combination according to one of the two protocols in Part B, the relay node N_3 broadcasts the network coded packet. The data processing at the end node is similar to that in [24] and is illustrated below, using N_2 as an example. The received signal at the end node N_2 is given by

$$y_{3,2}^l[m] = h_{3,2}^l x_3^l[m] + w_2^l[m] \tag{21}$$

where $x_3^l[m]$ is given in (20). Since $v_3^l[m]$ in (20) is an estimate of $x_1^l[m] \cdot x_2^l[m]$, it can be regarded as $x_1^l[m] \cdot x_2^l[m]$ plus a virtual noise $w_v^l[m]$. In order to remove its self-information contained in $v_3^l[m]$, N_2 performs the following operation

$$\begin{aligned} \hat{y}_{3,2}^l[m] &= x_2^l[m] \cdot y_{3,2}^l[m] \\ &= h_{3,2}^l x_2^l[m] \cdot \alpha \left(x_1^l[m] \cdot x_2^l[m] + w_v^l[m] \right) + \hat{w}_2^l[m]. \\ &= h_{3,2}^l \alpha \left(x_1^l[m] + \hat{w}_v^l[m] \right) + \hat{w}_2^l[m] \end{aligned} \tag{22}$$

N_2 can then combine $\hat{y}_{3,2}^l[m]$ and $y_{1,2}^l[m]$ (the signal sent through the direct link between N_1 and N_2) with MRC. Finally, the original information packet U_1 sent from N_1 can be retrieved after the channel decoding.

IV. PERFORMANCE ANALYSIS OF SOFT NETWORK CODING

In this section, we analyze the performance of SoftNC in TWRC by investigating its ergodic capacity and outage capacity.

We first define the following notations. At node N_1, the instantaneous mutual information of our SoftNC system is $\mathcal{I}_{(h_{1,2},h_{1,3},h_{2,3},h_{3,2})}\left(x_1; y_{3,2}, y_{1,2}\right)$. It denotes how much information node N_2 can obtain from the received signals $y_{3,2}$ and $y_{1,2}$ given $h_{1,2}$, $h_{1,3}$, $h_{2,3}$, and $h_{3,2}$. Likewise, the instantaneous mutual information $\mathcal{I}_{(h_{2,1},h_{1,3},h_{2,3},h_{3,1})}\left(x_2; y_{3,1}, y_{2,1}\right)$ denotes how much information node N_1 can obtain from the received signals $y_{3,1}$ and $y_{2,1}$ given $h_{2,1}$, $h_{1,3}$, $h_{2,3}$, and $h_{3,1}$. Note that the subscript in $\mathcal{I}_{(a)}(\cdot)$ denotes the fact that $\mathcal{I}_{(a)}(\cdot)$ is a function of *a*.

The ergodic capacity, $\mathcal{R}_{\text{EC}}$, is defined as the ergodic mutual information that can be reliably transmitted between the two end nodes in the whole relay process. First, the ergodic mutual information between the two ends is

$$\mathcal{R}_{\text{EC}} = \underset{h_{i,j}}{E}\left\{ \mathcal{I}_{(h_{1,2},h_{1,3},h_{2,3},h_{3,2})}\left(x_1; y_{3,2}, y_{1,2}\right) + \mathcal{I}_{(h_{2,1},h_{1,3},h_{2,3},h_{3,1})}\left(x_2; y_{3,1}, y_{2,1}\right) \right\} \tag{23}$$

As shown in [25], the ergodic capacity in (23) is approachable when the capacity achieving channel coding spans a number of coherent periods ($L \gg 1$) to average out both the Gaussian noise and the fluctuations of the channel. That is

$$\mathcal{R}_{\text{EC}} = \lim_{L\to\infty} \sum_{l=1}^{L} \left(\mathcal{I}_{\left(h_{1,2}^l, h_{1,3}^l, h_{2,3}^l, h_{3,2}^l\right)} \left(x_1; y_{3,2}, y_{1,2} \right) + \mathcal{I}_{\left(h_{2,1}^l, h_{1,3}^l, h_{2,3}^l, h_{3,1}^l\right)} \left(x_2; y_{3,1}, y_{2,1} \right) \right). \tag{24}$$

The outage capacity, $\mathcal{R}_{\text{OC}}$ is defined as the maximum rate that can be transmitted between the two end nodes in the whole relay process under a given outage probability, when the whole network experiences slow fading with the channel coherence time being much larger than the codeword. That is

$$\begin{aligned} \mathcal{R}_{\text{OC}} &= \max \left\{ \mathcal{R}_1 + \mathcal{R}_2 \middle| \mathcal{P}\left(\mathcal{I}_{\left(h_{1,2}, h_{1,3}, h_{2,3}, h_{3,2}\right)} \left(x_1; y_{3,2}, y_{1,2} \right) \le \mathcal{R}_1 \right) \le \varepsilon, \mathcal{P}\left(\mathcal{I}_{\left(h_{2,1}, h_{1,3}, h_{2,3}, h_{3,1}\right)} \left(x_2; y_{3,1}, y_{2,1} \right) \le \mathcal{R}_2 \right) \le \varepsilon \right\} \\ &= \max \left\{ \mathcal{R}_1 \middle| \mathcal{P}\left(\mathcal{I}_{\left(h_{1,2}, h_{1,3}, h_{2,3}, h_{3,2}\right)} \left(x_1; y_{3,2}, y_{1,2} \right) \le \mathcal{R}_1 \right) \le \varepsilon \right\} + \max \left\{ \mathcal{R}_2 \middle| \mathcal{P}\left(\mathcal{I}_{\left(h_{2,1}, h_{1,3}, h_{2,3}, h_{3,1}\right)} \left(x_2; y_{3,1}, y_{2,1} \right) \le \mathcal{R}_2 \right) \le \varepsilon \right\} \end{aligned} \tag{25}$$

where ε is the given outage probability.

To find the ergodic capacity and outage capacity of SoftNC system, we first analyze the instantaneous mutual information in three steps. In the first two steps, we calculate the mutual information between the two end nodes with the help of SoftNC at the relay node N_3. Since it has been shown in Section III that it is the information of the network coded bits $d_1 \oplus d_2$ (or equivalently $x_1 \cdot x_2$) that node N_3 forwards, and that the forwarded signal x_3 in SoftNC can be seen as an estimate of $x_1 \cdot x_2$ based on the received signals $y_{1,3}$ and $y_{2,3}$, the information of $d_1 \oplus d_2$ contained in the forwarded signal x_3 can be calculated by assuming a virtual channel whose input is $x_1 \cdot x_2$ and the output is x_3. As shown in Fig. 4, in the first step, we calculate the mutual information between the virtual input $x_1 \cdot x_2$ and the forwarded signal x_3. That is

$$\mathcal{I}_{\text{step1}} \triangleq \mathcal{I}_{\left(h_{1,3}, h_{2,3}\right)} \left(x_3; \left(d_1 \oplus d_2 \right) \right) = \mathcal{I}_{\left(h_{1,3}, h_{2,3}\right)} \left(x_3; \left(x_1 \cdot x_2 \right) \right). \tag{26}$$

Note that $\mathcal{I}_{\text{step1}}$ is a function of the channel coefficients $h_{1,3}$ and $h_{2,3}$.

Based on the result in (26), in the second step, we then calculate the total information which can be exchanged between the two end nodes via the relay node N_3, which is defined as

$$\mathcal{I}_{\text{step2}} \triangleq \mathcal{I}_{\left(h_{1,3}, h_{2,3}, h_{3,2}\right)} \left(x_1; y_{3,2} \right) + \mathcal{I}_{\left(h_{1,3}, h_{2,3}, h_{3,1}\right)} \left(x_2; y_{3,1} \right). \tag{27}$$

Finally, in the third step, the total instantaneous mutual information $\mathcal{R}(h_{1,2}, h_{1,3}, h_{3,2}, h_{3,1})$ can be obtained by combining the result in the second step and the information that can be exchanged in the direct link between the two end nodes.

A. Analysis of AF-SoftNC

Step 1: Bounds of $\mathcal{I}_{\text{step1}}$

As shown in (16), the output of AF-SoftNC, x_3, is the LLR of $x_1 \cdot x_2$ based on the received signals $y_{1,3}$ and $y_{2,3}$. It turns out that the distribution of x_3 is very complicated and it is hard to calculate $\mathcal{I}_{\text{step1}}$ directly. However, an upper bound and a lower bound of $\mathcal{I}_{\text{step1}}$ can be found.

Theorem 1: For two Gaussian channels with any given coefficients $h_{1,3}$ and $h_{2,3}$, the mutual information $\mathcal{I}_{\text{step1}}$ in the AF-SoftNC and SBF-SoftNC protocols is bounded by

$$C_{\text{BPSK}}\left(h_{1,3}\right) \cdot C_{\text{BPSK}}\left(h_{2,3}\right) \le \mathcal{I}_{\text{step1}} \le \mathcal{G}\left(C_{\text{BPSK}}\left(h_{1,3}\right), C_{\text{BPSK}}\left(h_{2,3}\right)\right) \tag{28}$$

where $C_{\text{BPSK}}\left(h\right) \triangleq \mathcal{I}\left(x;\left(y = hx + w\right)\right)$ is the mutual information of a Gaussian channel with BPSK modulated input x, channel coefficient h, and Gaussian noise $w \sim \mathcal{N}\left(0,1\right)$. Its numerical calculation can be referred to the *Appendix* of [26]. The function $\mathcal{G}\left(x, y\right)$ in (28) is defined as

$$\mathcal{G}\left(x, y\right) \triangleq 1 - H\left(H^{-1}\left(x\right)\left(1 - H^{-1}\left(y\right)\right) + H^{-1}\left(y\right)\left(1 - H^{-1}\left(x\right)\right)\right) \tag{29}$$

where $H\left(p\right) \triangleq -p \log p - \left(1-p\right)\log\left(1-p\right)$ is the entropy of binary distribution with probability p and $H^{-1}(\cdot)$ is the inverse function of $H\left(\cdot\right)$.

The proof of this theorem is given in *Appendix I*, where it is shown that *Theorem 1* can be seen as a corollary of *Theorem 3* in [27]. According to the results in [27] and our extensive simulations, the two bounds are close to each other and both of them can serve as good approximations to $\mathcal{I}_{\text{step1}}$. Some simulation result about (28) will be given in Section V.

Step 2: Approximation of $\mathcal{I}_{\text{step2}}$

As shown in (27), $\mathcal{I}_{\text{step2}}$ is defined as the total information that can be exchanged between the two end nodes via the relay node N_3. Since in TWRC each end node knows its self-information, we have

$$\begin{aligned} \mathcal{I}_{\text{step2}} &\triangleq \mathcal{I}_{\left(h_{1,3},h_{2,3},h_{3,2}\right)}\left(x_1; y_{3,2}\right) + \mathcal{I}_{\left(h_{1,3},h_{2,3},h_{3,1}\right)}\left(x_2; y_{3,1}\right) \\ &= \mathcal{I}_{\left(h_{1,3},h_{2,3},h_{3,2}\right)}\left(\left(x_1 \cdot x_2\right); y_{3,2}\right) + \mathcal{I}_{\left(h_{1,3},h_{2,3},h_{3,1}\right)}\left(\left(x_1 \cdot x_2\right); y_{3,1}\right). \end{aligned} \tag{30}$$

In the following, we consider $\mathcal{I}_{(h_{1,3},h_{2,3},h_{3,2})}((x_1 \cdot x_2); y_{3,2})$ as an example, and $\mathcal{I}_{(h_{1,3},h_{2,3},h_{3,1})}((x_1 \cdot x_2); y_{3,1})$ can be calculated accordingly. It turns out that the distribution of $y_{3,2} = h_{3,2}x_3 + w_{3,2}$ is difficult to obtain due to the complicated distribution of v_3. Fortunately, v_3 given in (16) has exactly the same expression as the output LLR from a check node in the widely used sum-product decoding algorithm [21]. As it is shown in [28], a random variable with such an expression, although tends to be less like Gaussian, can be well approximated by the Gaussian distribution. Motivated by this result, we approximate v_3 as

$$v_3 \approx v_3' \triangleq h_v (x_1 \cdot x_2) + w_v \tag{31}$$

where w_v is a Gaussian noise with the distribution $w_v \sim \mathcal{N}(0,1)$ and h_v is a variable (real) which will be explained later. Then the forwarded signal can be approximated by

$$x_3 \approx x_3' \triangleq \beta v_3' = \beta\left(h_v (x_1 \cdot x_2) + w_v\right) \qquad \text{with } E\left\{|x_3'|^2\right\} = 1 \tag{32}$$

where $\beta = 1\big/\sqrt{h_v^2 + 1}$ is a normalizing factor in order to satisfy the power constraint $E\left\{|x_3'|^2\right\} = 1$. As shown in Step 1 and Fig. 4, v_3 can be seen as the output of a virtual channel whose input is $x_1 \cdot x_2$. The approximation in (31) further simplifies the virtual channel as a Gaussian channel with a real channel coefficient h_v and an additive Gaussian noise w_v, as shown in Fig. 5. Based on this approximation, we have

$$C_{\text{BPSK}}(h_v) \triangleq \mathcal{I}_{(h_v)}((x_1 \cdot x_2); x_3') \approx \mathcal{I}_{(h_{1,3},h_{2,3})}((x_1 \cdot x_2); x_3). \tag{33}$$

Since it is shown in Step one that the bounds of $\mathcal{I}_{\text{step1}}$, which is given in *Theorem 1*, is very close to the true mutual information, we take the upper bound as the approximation of $\mathcal{I}_{\text{step1}}$. That is

$$\mathcal{I}_{\text{step1}} \triangleq \mathcal{I}_{(h_{1,3},h_{2,3})}((x_1 \cdot x_2); x_3) \approx \mathcal{G}\left(C_{\text{BPSK}}(h_{1,3}), C_{\text{BPSK}}(h_{2,3})\right). \tag{34}$$

By considering (33) and (34) together, we have the channel coefficient h_v as

$$h_v = C_{\text{BPSK}}^{-1}\left(\mathcal{G}\left(C_{\text{BPSK}}(h_{1,3}), C_{\text{BPSK}}(h_{2,3})\right)\right) \tag{35}$$

where $C_{\text{BPSK}}^{-1}(\cdot)$ is the reverse function of $C_{\text{BPSK}}(\cdot)$ and its numerical calculation can be referred to [26]. According to (31) and as shown in Fig. 5, the received signal at node N_2 is then given by

$$y_{3,2} \approx y_{3,2}' = h_{3,2}x_3' + w_{3,2} = h_{3,2}\beta h_v (x_1 \cdot x_2) + h_{3,2}\beta w_v + w_{3,2}. \tag{36}$$

Given channel coefficients $h_{1,3}$ and $h_{3,2}$, we have

$$\begin{aligned}\mathcal{I}_{(h_{1,3},h_{2,3},h_{3,2})}\left(x_1;y_{3,2}\right)&=\mathcal{I}_{(h_{1,3},h_{2,3},h_{3,2})}\left(\left(x_1\cdot x_2\right);y_{3,2}\right)\\&\approx\mathcal{I}_{(h_v,h_{3,2})}\left(\left(x_1\cdot x_2\right);y'_{3,2}\right)=C_{\mathrm{BPSK}}\left(\sqrt{\frac{h_v{}^2\left|h_{3,2}\right|^2}{h_v^2+\left|h_{3,2}\right|^2+1}}\right).\end{aligned}\tag{37}$$

$\mathcal{I}_{(h_{1,3},h_{2,3},h_{3,1})}\left(x_2;y_{3,1}\right)$ can be approximated by using the same procedure in the approximation of $\mathcal{I}_{(h_{1,3},h_{2,3},h_{3,2})}\left(x_1;y_{3,2}\right)$. Finally by substituting these results into (30), we have

$$\mathcal{I}_{\mathrm{step2}}\approx C_{\mathrm{BPSK}}\left(\sqrt{\frac{h_v{}^2\left|h_{3,2}\right|^2}{h_v^2+\left|h_{3,2}\right|^2+1}}\right)+C_{\mathrm{BPSK}}\left(\sqrt{\frac{h_v{}^2\left|h_{3,1}\right|^2}{h_v^2+\left|h_{3,1}\right|^2+1}}\right).\tag{38}$$

Again, based on the extensive simulations, it is found that the approximation in (38) is close to the true mutual information. Some selected simulation result is also presented and verifies the accuracy of (38) in Section V.

Step 3: Approximation of instantaneous mutual information

As defined in (24), $\mathcal{R}(h_{1,2},h_{1,3},h_{3,2},h_{3,1})$ is the total *instantaneous* mutual information between the two end nodes through both the relay link and the direct link. Take node N_2. It receives $y_{1,2}$ from node N_1 (direct link) in the first time slot and $y_{3,2}$ from node N_3 (relay link) in the third time slot. Due to the fact that node N_2 knows its self-information x_2, and also based on the good approximation in (36), x_2 can be removed from $y_{3,2}$ by

$$\widetilde{y_{3,2}}\triangleq y_{3,2}\cdot x_2\approx y'_{3,2}\cdot x_2=h_{3,2}\beta h_v x_1+h_{3,2}\beta w_v+w_{3,2}\,.\tag{39}$$

Assume that node N_2 has the channel state information of all the three channels ($h_{1,3}$, $h_{1,2}$, and $h_{3,2}$), it can perform MRC to $\widetilde{y_{3,2}}$ and $y_{1,2}$, whose SNR values are $\left(h_v{}^2\left|h_{3,2}\right|^2\right)\Big/\left(h_v^2+\left|h_{3,2}\right|^2+1\right)$ and $|h_{1,2}|^2$, respectively. As a result, the final SNR of the combined signal can be approximated as $\left(h_v{}^2\left|h_{3,2}\right|^2\right)\Big/\left(h_v^2+\left|h_{3,2}\right|^2+1\right)+\left|h_{1,2}\right|^2$. Then, the total information of x_1 obtained at node N_2 through both the direct link and the relay link can be approximated as

$$\mathcal{I}_{(h_{1,2},h_{1,3},h_{2,3},h_{3,2})}\left(x_1;y_{3,2},y_{1,2}\right)\approx\mathcal{I}_{(h_{1,2},h_{3,2},h_v)}\left(x_1;\widetilde{y_{3,2}},y_{1,2}\right)=C_{\mathrm{BPSK}}\left(\sqrt{\frac{h_v{}^2\left|h_{3,2}\right|^2}{h_v^2+\left|h_{3,2}\right|^2+1}+\left|h_{1,2}\right|^2}\right).\tag{40}$$

By the same procedure, we can obtain the approximation of $\mathcal{I}_{(h_{1,2},h_{1,3},h_{2,3},h_{3,1})}\left(x_2;y_{3,1},y_{2,1}\right)$.

Step 4: Approximation of ergodic capacity and outage capacity

By averaging over all channel fading coefficients, the ergodic mutual information rate of AF-SoftNC can be approximated by

$$\mathcal{R}_{\text{EC,AF-SoftNC}} \approx \underset{h_{i,j}}{E}\left\{C_{\text{BPSK}}\left(\sqrt{\frac{h_v{}^2\left|h_{3,2}\right|^2}{h_v^2+\left|h_{3,2}\right|^2+1}+\left|h_{1,2}\right|^2}\right)+C_{\text{BPSK}}\left(\sqrt{\frac{h_v{}^2\left|h_{3,1}\right|^2}{h_v^2+\left|h_{3,1}\right|^2+1}+\left|h_{2,1}\right|^2}\right)\right\}. \tag{41}$$

We now consider the outage capacity by taking N_1 as an example. Since $C_{BPSK}(\bullet)$ is a monotonic increasing function, the outage probability of N_1 can be expressed as

$$\begin{aligned}\varepsilon &= \mathcal{P}\left(\mathcal{I}_{\left(h_{1,2},h_{1,3},h_{2,3},h_{3,2}\right)}\left(x_1;y_{3,2},y_{1,2}\right)\le\mathcal{R}_1\right)\\ &\approx \mathcal{P}\left(C_{\text{BPSK}}\left(\sqrt{\frac{h_v{}^2\left|h_{3,2}\right|^2}{h_v^2+\left|h_{3,2}\right|^2+1}+\left|h_{1,2}\right|^2}\right)\le\mathcal{R}_1\right)=\mathcal{P}\left(\sqrt{\frac{h_v{}^2\left|h_{3,2}\right|^2}{h_v^2+\left|h_{3,2}\right|^2+1}+\left|h_{1,2}\right|^2}\le C_{\text{BPSK}}^{-1}\left(\mathcal{R}_1\right)\right)\end{aligned} \tag{42}$$

Given the Rayleigh distribution of $h_{i,j}$, we could obtain the complementary cumulative distribution function of $\sqrt{\frac{h_v{}^2\left|h_{3,2}\right|^2}{h_v^2+\left|h_{3,2}\right|^2+1}+\left|h_{1,2}\right|^2}$ (maybe in numerical method), denoted by $F_1(\bullet)$ [6]. Similar to [Ch5, 25], we can obtain the outage capacity of $\mathcal{R}_1$ as

$$\mathcal{R}_1 = C_{\text{BPSK}}\left(F_1^{-1}(\varepsilon)\right) \tag{43}$$

In a similar way, $\mathcal{R}_2 = C_{\text{BPSK}}\left(F_2{}^{-1}(\varepsilon)\right)$ can be obtained, where $F_2(\bullet)$ is the complementary cumulative distribution function of $\sqrt{\frac{h_v{}^2\left|h_{3,1}\right|^2}{h_v^2+\left|h_{3,1}\right|^2+1}+\left|h_{1,1}\right|^2}$. The outage capacity of our AF-SoftNC system is approximated by

$$\mathcal{R}_{\text{OC,AF-SoftNC}} \approx C_{\text{BPSK}}\left(F_1^{-1}(\varepsilon)\right)+C_{\text{BPSK}}\left(F_2{}^{-1}(\varepsilon)\right) \tag{44}$$

B. Analysis of SBF-SoftNC

The analysis procedure for SBF-SoftNC is similar to that for AF-SoftNC protocol and is also taken in three steps. As shown in *Theorem 1*, the approximation of $\mathcal{I}_{\text{step1}}$ is the same in AF-SoftNC and SBF-SoftNC. We now discuss the following two steps in the analysis of SBF-SoftNC.

Step 2: Approximation of $\mathcal{I}_{\text{step2}}$

Since each end node knows its self-information, (30) still holds for SBF-SoftNC. As shown in (14), the soft bit generated at the relay node is in fact the "tanh" function of the LLR value. Since the LLR of

[6] It turns out that it is very difficult to obtain an analytical expression of $F_1(\bullet)$. However, it can be evaluated numerically.

$x_1 \cdot x_2$ can be approximated by $h_v\left(x_1 \cdot x_2\right)+w_v$ as in (31), it is reasonable to use a new random variable $\tanh\left(h_v\left(x_1 \cdot x_2\right)+w_v\right)$ to approximate the soft bit of $x_1 \cdot x_2$, i.e., v_3 in SBF-SoftNC. That is

$$v_3 \approx v_3'' \triangleq \tanh\left(h_v\left(x_1 \cdot x_2\right)+w_v\right) \tag{45}$$

where h_v and w_v have the same meaning as (31). Similar to (32), the forwarded signal can then be approximated by

$$x_3 \approx x_3'' \triangleq \theta \tanh\left(h_v\left(x_1 \cdot x_2\right)+w_v\right) \qquad \text{with } E\left\{\left|x_3''\right|^2\right\}=1 \tag{46}$$

where θ is a normalizing factor to satisfy the constraint $E\left\{\left|x_3''\right|^2\right\}=1$. It can be seen from (45) that in the case of SBF-SoftNC the virtual channel in Fig. 4 can be approximated with the Gaussian channel followed by a soft bit transform ("tanh" function), as shown in Fig. 6. As a result, the received signal at the end node N_2, $y_{3,2}$, can be approximated by

$$y_{3,2} \approx y_{3,2}'' = h_{3,2}\theta \tanh\left(h_v\left(x_1 \cdot x_2\right)+w_v\right)+w_{3,2}\,. \tag{47}$$

However, it is still difficult to calculate the mutual information between $x_1 \cdot x_2$ and $y_{3,2}''$, i.e., $\mathcal{I}\left(\left(x_1 \cdot x_2\right); y_{3,2}''\right)$. According to the conclusion of the capacity investigation of the memeoryless relay protocols in OWRC [22], the AF protocol and the demodulate-and-forward (i.e., forward $\operatorname{sign}\left(\tilde{y}_{1,3}^{l}[m]\right)$ in (12) to the destination node) protocol can approach the maximum mutual information performance (the mutual information of the SBF protocol) in low SNR region and high SNR region respectively. By generalizing this result into the TWRC case, we further approximate $\mathcal{I}\left(\left(x_1 \cdot x_2\right); y_{3,2}''\right)$ by

$$\mathcal{I}\left(\left(x_1 \cdot x_2\right); y_{3,2}''\right)=\max\left\{\mathcal{I}_{\left(h_{3,2},h_v\right)}\left(\left(x_1 \cdot x_2\right); y_{3,2}'\right), \mathcal{I}_{\left(h_{3,2},h_v\right)}\left(\left(x_1 \cdot x_2\right); y_{3,2}^{\text{Hard}}\right)\right\} \tag{48}$$

where $y_{3,2}^{\text{Hard}} \triangleq h_{3,2}\operatorname{sign}\left(h_v\left(x_1 \cdot x_2\right)+w_v\right)+w_{3,2}$ and $\mathcal{I}_{\left(h_{3,2},h_v\right)}\left(\left(x_1 \cdot x_2\right); y_{3,2}'\right)$ is given in (37). The calculation of $\mathcal{I}_{\left(h_{3,2},h_v\right)}\left(\left(x_1 \cdot x_2\right); y_{3,2}^{\text{Hard}}\right)$ is given in *Appendix II*. Finally, we obtain

$$\begin{aligned}\mathcal{I}_{\left(h_{1,3},h_{2,3},h_{3,2}\right)}\left(x_1; y_{3,2}\right) &= \mathcal{I}_{\left(h_{1,3},h_{2,3},h_{3,2}\right)}\left(\left(x_1 \cdot x_2\right); y_{3,2}\right) \\ &\approx \mathcal{I}_{\left(h_v,h_{3,2}\right)}\left(\left(x_1 \cdot x_2\right); y_{3,2}''\right) \\ &\approx \max\left\{C_{\text{BPSK}}\left(\sqrt{\frac{h_v^{\,2}\,|\,h_{3,2}\,|^2}{h_v^{\,2}+|\,h_{3,2}\,|^2+1}}\right), \mathcal{G}\left(C_{\text{BPSK}}^{\text{Hard}}\left(h_v\right), C_{\text{BPSK}}\left(h_{3,2}\right)\right)\right\}\end{aligned}\,. \tag{49}$$

$\mathcal{I}_{\left(h_{1,3},h_{2,3},h_{3,1}\right)}\left(x_2; y_{3,1}\right)$ can be found in the same way. Finally, $\mathcal{I}_{\text{step2}}$ can be approximated by

$$\begin{aligned}\mathcal{I}_{\text{step2}} \approx & \max\left\{C_{\text{BPSK}}\left(\sqrt{\frac{h_v^{\,2}\,|\,h_{3,2}\,|^2}{h_v^{\,2}+|\,h_{3,2}\,|^2+1}}\right),\mathcal{G}\left(C_{\text{BPSK}}^{\text{Hard}}\left(h_v\right),C_{\text{BPSK}}\left(h_{3,2}\right)\right)\right\} \\ & +\max\left\{C_{\text{BPSK}}\left(\sqrt{\frac{h_v^{\,2}\,|\,h_{3,1}\,|^2}{h_v^{\,2}+|\,h_{3,1}\,|^2+1}}\right),\mathcal{G}\left(C_{\text{BPSK}}^{\text{Hard}}\left(h_v\right),C_{\text{BPSK}}\left(h_{3,1}\right)\right)\right\}\end{aligned}. \tag{50}$$

Based on the extensive simulations, it is found that the approximation in (50) is close to the true mutual information. Some simulation result to verify the accuracy of (50) is also presented in Section V.

Step 3: Approximation of instantaneous mutual information

Since v_3 in (45) is the approximation of the soft bit of $x_1 \cdot x_2$, i.e. its MMSE estimation, it can also be regarded as $x_1 \cdot x_2$ plus certain noise. Then, similar to the approximation in Part A, the node N_2 first removes its self-information from the received signal $y_{3,2}$ as follows

$$\widetilde{y''_{3,2}} = y_{3,2} \cdot x_2 \approx y''_{3,2} \cdot x_2 \approx h''_{v,2} x_1 + w''_{v,2} \tag{51}$$

where the real variable $h''_{v,2}$ is the virtual channel coefficient and $w''_{v,2}$ includes the noise $w_{3,2}$ introduced at N_2 and the equivalent noise introduced at the relay node and is approximated to be Gaussian distributed, i.e., $w''_{v,2} \sim \mathcal{N}\left(0,1\right)$. Then the virtual channel coefficient $h''_{v,2}$ can be determined by

$$\begin{aligned}h''_{v,2} &= C_{\text{BPSK}}^{-1}\left(\mathcal{I}_{h_{1,3},h_{2,3},h_{3,2}}\left(x_1; y_{3,2}\right)\right) \approx C_{\text{BPSK}}^{-1}\left(\mathcal{I}_{h_v,h_{3,2}}\left(\left(x_1 \cdot x_2\right); y''_{3,2}\right)\right) \\ &\approx \max\left\{\sqrt{\frac{h_v^{\,2}\left|h_{3,2}\right|^2}{h_v^{\,2}+\left|h_{3,2}\right|^2+1}}, C_{\text{BPSK}}^{-1}\left(\mathcal{G}\left(C_{\text{BPSK}}^{\text{Hard}}\left(h_v\right),C_{\text{BPSK}}\left(h_{3,2}\right)\right)\right)\right\}\end{aligned}. \tag{52}$$

After combining $\widetilde{y''_{3,2}}$ and $y_{1,2}$, the SNR of the new signal is $\left(h''_{v,2}\right)^2+\left|h_{1,2}\right|^2$. The total information of x_1 obtained at node N_2 through both the direct link and the relay link can be approximated as

$$\mathcal{I}\left(x_1; y_{3,2}, y_{1,2}\right) \approx \mathcal{I}\left(x_1; \widetilde{y''_{3,2}}, y_{1,2}\right) \approx C_{\text{BPSK}}\left(\sqrt{\left(h''_{v,2}\right)^2+\left|h_{1,2}\right|^2}\right). \tag{53}$$

Note that in (51) we assume that $w''_{v,2} \sim \mathcal{N}\left(0,1\right)$ in order to combine $\widetilde{y''_{3,2}}$ and $y_{1,2}$ with MRC. We argue that different assumptions on the distribution of $w''_{v,2}$ do not affect the final result in (53) too much because, based on the conclusion in [27], the mutual information between x_1 and the two binary input symmetric memoryless output (BISMO) channels' outputs, $\widetilde{y''_{3,2}}$ and $y_{1,2}$, does not depend too much on the distribution of each channel when $\mathcal{I}(x_1; y_{1,2})$ and $\mathcal{I}(x_1; \widetilde{y''_{3,2}})$ are given.

In a similar way, we can obtain the approximation $\mathcal{I}\left(x_2; y_{3,1}, y_{2,1}\right) \approx C_{\text{BPSK}}\left(\sqrt{\left(h''_{v,1}\right)^2 + \left|h_{2,1}\right|^2}\right)$ where the virtual channel coefficient $h''_{v,1}$ is similar to (52)

$$\begin{aligned} h''_{v,1} &= C_{\text{BPSK}}^{-1}\left(\mathcal{I}_{h_{1,3},h_{2,3},h_{3,1}}\left(x_2; y_{3,1}\right)\right) \approx C_{\text{BPSK}}^{-1}\left(\mathcal{I}_{h_v,h_{3,1}}\left(\left(x_1 \cdot x_2\right); y''_{3,1}\right)\right) \\ &\approx \max\left\{\sqrt{\frac{h_v{}^2\left|h_{3,1}\right|^2}{h_v{}^2 + \left|h_{3,1}\right|^2 + 1}}, C_{\text{BPSK}}^{-1}\left(\mathcal{G}\left(C_{\text{BPSK}}^{\text{Hard}}\left(h_v\right), C_{\text{BPSK}}\left(h_{3,1}\right)\right)\right)\right\}. \end{aligned} \tag{54}$$

Step 4: Approximation of ergodic capacity and outage capacity

By averaging over all the channel realizations, we can obtain the approximation of ergodic capacity for the SBF-SoftNC as

$$\mathcal{R}_{\text{EC, SBF-SoftNC}} \approx \underset{h_{i,j}}{E}\left\{C_{\text{BPSK}}\left(\sqrt{\left(h''_{v,2}\right)^2 + \left|h_{1,2}\right|^2}\right)\right\} + \underset{h_{i,j}}{E}\left\{C_{\text{BPSK}}\left(\sqrt{\left(h''_{v,1}\right)^2 + \left|h_{2,1}\right|^2}\right)\right\}. \tag{55}$$

By denoting the complementary cumulative distribution function of $\sqrt{\left(h''_{v,2}\right)^2 + \left|h_{1,2}\right|^2}$ and $\sqrt{\left(h''_{v,1}\right)^2 + \left|h_{2,1}\right|^2}$ with $F_3(\bullet)$ and $F_4(\bullet)$ respectively, we can obtain the approximation of outage capacity for the SBF-SoftNC as

$$\mathcal{R}_{\text{OC,AF-SoftNC}} \approx C_{\text{BPSK}}\left(F_3^{-1}(\varepsilon)\right) + C_{\text{BPSK}}\left(F_4^{-1}(\varepsilon)\right) \tag{56}$$

V. Simulation and Performance Comparison

In this section, we first present simulation results to show the accuracy of the bounds and approximations which have been derived in Section IV in the first and second steps of the instantaneous mutual information analysis. We then compare the ergodic capacity and outage capacity performance of the proposed AF-SoftNC and SBF-SoftNC with that of the conventional relay protocols in TWRC.

A. Simulation results of bounds and approximations

We first investigate the accuracy of the upper and lower bounds for $\mathcal{I}_{\text{step1}}$, given in (28) in Theorem 1. The performance of the theoretical bounds and that of the true mutual information in AF-SoftNC and SBF-SoftNC designs are compared in Fig. 7 with different channel gains $\gamma_{1,3} \triangleq \left|h_{1,3}\right|^2$ and $\gamma_{2,3} \triangleq \left|h_{2,3}\right|^2$. In the simulation of this subsection, we assume $\gamma_{i,j} = \gamma_{j,i}\ \forall i, j \in \{1,2,3\}$. Two groups of curves are presented, where $\gamma_{1,3}$ is set to 0dB for one group and is set to be the same as $\gamma_{2,3}$ for the other group. The true mutual information of $\mathcal{I}_{\text{step1}}$ for AF-SoftNC and SBF-SoftNC is calculated according to its definition. That is

$$\begin{aligned}\mathcal{I}\left(\left(x_1 \cdot x_2\right); x_3\right) &= \mathcal{H}\left(x_3\right) - \mathcal{H}\left(x_3 \left|\left(x_1 \cdot x_2\right)\right.\right) \\ &= \mathcal{H}\left(x_3\right) - \mathcal{H}\left(x_3 \left|\left(x_1 = x_2 = \pm 1\right)\right.\right)\end{aligned}. \tag{57}$$

As shown in Fig. 7, both the upper and the lower bounds are close to $\mathcal{I}_{\text{step1}}$ and can be seen as good approximations in the analysis of AF-SoftNC and SBF-SoftNC systems.

Next, in Fig. 8, we present some simulation results in the approximation of $\mathcal{I}_{\text{step2}}$ with different $\gamma_{1,3}$ and $\gamma_{2,3}$. In the simulation, $\gamma_{2,3} = \gamma_{3,2}$ is set to 0dB for one group of curves and is set to the same as $\gamma_{1,3} = \gamma_{3,1} = \gamma_{3,2} = \gamma_{2,3}$ for the other group. The curves of the approximation results for AF-SoftNC and SBF-SoftNC are obtained according to (38) and (50), respectively. The true mutual information is given by the definition

$$\begin{aligned}\mathcal{I}\left(x_1; y_{3,2}\right) + \mathcal{I}\left(x_2; y_{3,1}\right) &= \mathcal{I}\left(\left(x_1 \cdot x_2\right); y_{3,2}\right) + \mathcal{I}\left(\left(x_1 \cdot x_2\right); y_{3,1}\right) \\ &= \mathcal{H}\left(y_{3,2}\right) - \mathcal{H}\left(y_{3,2} \left| x_1 = x_2\right.\right) + \mathcal{H}\left(y_{3,1}\right) - \mathcal{H}\left(y_{3,1} \left| x_1 = x_2\right.\right)\end{aligned} \tag{58}$$

where $\mathcal{H}\left(x\right)$ denotes the entropy of a random variable *x*.

It is shown in Fig. 8 that the proposed approximation results are close to $\mathcal{I}_{\text{step2}}$ for almost the whole range of SNR values. It is also shown that the SBF-SoftNC has a better performance than the AF-SoftNC design especially when the SNR is high, which has also been shown in (50). This result is consistent with that in the comparison of AF and SBF in OWRC [22].

B. Ergodic capacity and outage capacity

In this part, we compare the ergodic capacity and outage capacity performance of AF-SoftNC and SBF-SoftNC with the traditional network coding based two-way relay protocol. This protocol is similar to the SNC protocol in [4], where the relay node decodes the two received packets from the two end nodes separately in an explicit manner, and then combines the packets into a network coded packet for forwarding.The ergodic capacity of the traditional straightforward network coding is

$$\begin{aligned}\mathcal{R}_{\text{EC, SNC}} = \min&\left\{\underset{h_{i,j}}{E}\left\{C_{\text{BPSK}}\left(\sqrt{\left|h_{1,2}\right|^2 + \left|h_{3,2}\right|^2}\right)\right\}, \underset{h_{i,j}}{E}\left\{C_{\text{BPSK}}\left(h_{1,3}\right)\right\}\right\} \\ + \min&\left\{\underset{h_{i,j}}{E}\left\{C_{\text{BPSK}}\left(\sqrt{\left|h_{2,1}\right|^2 + \left|h_{3,1}\right|^2}\right)\right\}, \underset{h_{i,j}}{E}\left\{C_{\text{BPSK}}\left(h_{2,3}\right)\right\}\right\}\end{aligned}. \tag{59}$$

Its outage capacity is simulated according to the definition as

$$\mathcal{R}_{\text{OC, SNC}} = \max\left\{\mathcal{R}_1 \middle| \mathcal{P}\left(\min\left\{C_{\text{BPSK}}\left(\sqrt{|h_{1,2}|^2+|h_{3,2}|^2}\right), C_{\text{BPSK}}\left(h_{1,3}\right)\right\} \le \mathcal{R}_1\right) \le \varepsilon\right\}$$
$$+\max\left\{\mathcal{R}_2 \middle| \mathcal{P}\left(\min\left\{C_{\text{BPSK}}\left(\sqrt{|h_{2,1}|^2+|h_{3,1}|^2}\right), C_{\text{BPSK}}\left(h_{2,3}\right)\right\} \le \mathcal{R}_2\right) \le \varepsilon\right\} \tag{60}$$

In Fig. 9, we compare the ergodic capacity performance of different two-way relay protocols when fixing the qualities (in terms of average SNR) of relay channels (the channels between the end nodes and the relay node) and varying the quality of direct channel (the channel between the two end nodes). We assume Rayleigh fading and symmetric average SNR $\lambda_{i,j} = \lambda_{j,i}\ \forall i, j \in \{1,2,3\}$. Specifically, the average SNR of the relay channels is set to 5dB, i.e. $\lambda_{3,1} = \lambda_{3,2} = \lambda_{1,3} = \lambda_{2,3} = \sqrt{10}$, and that of the direct channel $\lambda_{1,2}$ varies from -5dB to 15dB. In our simulation, “one channel use” means one transmission process of SoftNC (including three time slots). As shown Fig. 9, SoftNC schemes outperform the traditional SNC scheme in terms of ergodic capacity when the SNR of the direct channel is larger than 2dB.

In Fig. 10, we compare the outage capacity performance of different schemes under the same settings and the target outage probability is $\varepsilon = 0.05$. It is shown that our SoftNC schemes outperform the traditional SNC scheme for almost all the simulated SNR region. Notable is the fact that the SNC scheme involves more processing in terms of channel decoding and re-encoding at the relay. In both Fig. 9 and Fig. 10, the performance of the traditional SNC scheme is constant because it is constrained by the unchanged relay links.

In Fig. 11, we investigate the effect of the relay channel quality on the ergodic capacity performance of different two-way relay protocols, respectively. Specifically, in the simulation, the average channel SNR $\lambda_{2,3} = \lambda_{3,2} = 10\text{dB}$, $\lambda_{2,1} = \lambda_{1,2} = 0\text{dB}$ and the average SNR of the relay channel between N_1 and N_3, $\lambda_{1,3} = \lambda_{3,1} = -10$ dB to 30dB. As shown in Fig. 11, the ergodic capacities of the two proposed SoftNC schemes are better than the traditional SNC scheme except when $\gamma_{1,3} = \gamma_{3,1}$ falls in the range of 3dB to 23dB. This is because the transmission rate of SNC is severely constrained by $\gamma_{1,3}$ when it is much less $\gamma_{2,3}$ and the transmission rate of SNC is constrained to constant by $\gamma_{2,3}$ when $\gamma_{1,3}$ is larger than 23 dB.

In Fig. 12, we investigate the outage capacity performance under the same simulation settings as those in Fig. 11. Different from the results in Fig. 11, the outage capacities of the two proposed SoftNC schemes are larger than that of the transitional SNC scheme along the whole range of $\gamma_{1,3}$.

From the simulation, we can see that our proposed SoftNC schemes outperform the traditional SNC scheme in pretty general scenarios, especially in terms of outage capacity, even though SNC is much more complex than our schemes due to the channel decoding and re-encoding at the lay.

VI. CONCLUSION

In this paper, we propose two new network coding schemes for wireless TWRC, AF-SoftNC and SBF-SoftNC. In both schemes, the relay forwards "soft decision" information on the XOR of the two bits received from the two end nodes. In AF-SoftNC, the log-likelihood ratio is forwarded; in SBF-SoftNC, the MMSE estimate is forwarded. In this way, only symbol-level operation, rather than packet-level channel decoding and re-encoding operation (for example the traditional network coding), is required at the relay.

We analyze the ergodic capacity and the outage capacity of the SoftNC schemes. Compared to traditional network coding, our proposed SoftNC schemes perform better under some common scenarios, such as the case when the direct link has a better quality, although the relay is appreciable less complex. For implementation, in time-varying wireless networks, the relay node may adaptively select SoftNC or traditional SNC network coding based on channel quality and on its own capability (energy, hardware, and so on).

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APPENDIX I: PROOF OF THEOREM 1

Note that the multiplication of two BPSK symbols in the transmitted packet, $x_1 \cdot x_2$, is equivalent to the XOR of two binary bits in the corresponding codewords, $d_1 \oplus d_2$. Since Gaussian channel belongs to BISMO channels, the following inequalities can be obtained from *Theorem 2* in [27] directly

$$C_{bpsk}(h_{1,3})C_{bpsk}(h_{2,3}) \le \mathcal{I}\left((x_1 \cdot x_2); y_{1,3}, y_{2,3}\right) = \mathcal{I}\left((x_1 \cdot x_2); v_{1,3}, v_{2,3}\right) \le \mathcal{G}(C_{bpsk}(h_{1,3}), C_{bpsk}(h_{2,3})) \,. \quad \text{(A1)}$$

According to the signal processing (16) in AF-SoftNC and that (18) in SBF-SoftNC, it can be found that no information about $x_1 \cdot x_2$ is lost, i.e., $\mathcal{I}\left((x_1 \cdot x_2); y_{1,3}, y_{2,3}\right) = \mathcal{I}\left((x_1 \cdot x_2); x_3\right)$. We take the SBF-SoftNC protocol as an example to prove it

$$\begin{aligned}\mathcal{I}\left(\left(x_1\cdot x_2\right);y_{1,3},y_{2,3}\right)&=\mathcal{H}\left(x_1\cdot x_2\right)-\mathcal{H}\left(\left(x_1\cdot x_2\right)\middle|y_{1,3},y_{2,3}\right)\\&=1-H\left(\Pr\left(\left(x_1\cdot x_2\right)=-1\middle|y_{1,3},y_{2,3}\right)\right)\\&=1-H\left(\frac{1-\left(\Pr\left(\left(x_1\cdot x_2\right)=1\middle|y_{1,3},y_{2,3}\right)-\Pr\left(\left(x_1\cdot x_2\right)=-1\middle|y_{1,3},y_{2,3}\right)\right)}{2}\right)\\&=1-H\left(\frac{1-x_3}{2}\right)=1-H\left(\Pr\left(\left(x_1\cdot x_2\right)=-1\middle|x_3\right)\right)\\&=1-\mathcal{H}\left(\left(x_1\cdot x_2\right)\middle|x_3\right)=\mathcal{I}\left(\left(x_1\cdot x_2\right);x_3\right)\end{aligned}\tag{A2}$$

This completes the proof.

APPENDIX II: MUTUAL INFORMATION FOR DEMODULATE-AND-FORWARD

This appendix shows the approximation of $\mathcal{I}_{\left(h_{3,2},h_v\right)}\left(\left(x_1\cdot x_2\right);y_{3,2}^{\text{Hard}}\right)$ in (48), which can be seen as the mutual information between the source and destination of a two-hop memoryless relay channel, where $x_1\cdot x_2$ is the transmitted signal from the source, h_v and $h_{3,2}$ are the channel coefficients of the two hops, respectively, w_v and $w_{3,2}$ are the noises at relay node and destination node respectively. Since it is assumed that the demodulate-and-forward protocol is used at the relay, the mutual information of the first hop with hard decision is given by

$$\begin{aligned}\mathcal{I}\left(\left(x_1\cdot x_2\right);\operatorname{sign}\left(h_v\left(x_1\cdot x_2\right)+w_v\right)\right)&=1-H\left(\Pr\left(\left(x_1\cdot x_2\right)\neq\operatorname{sign}\left(h_v\left(x_1\cdot x_2\right)+w_v\right)\right)\right)\\&=1-H\left(1-\Phi_{0,1}\left(\left|h_v\right|\right)\right)\triangleq C_{\text{BPSK}}^{\text{Hard}}\left(h_v\right)\end{aligned}\tag{A3}$$

where $\Phi_{0,1}(\cdot)$ is the cumulative distribution function of a normal distribution $\mathcal{N}(0,1)$. It can be seen that the capacity of the second hop is $C_{\text{BPSK}}\left(h_{3,2}\right)$. Although it is difficult to calculate the exact formula of $\mathcal{I}_{\left(h_{3,2},h_v\right)}\left(\left(x_1\cdot x_2\right);y_{3,2}^{\text{Hard}}\right)$, the following theorem provides a tight upper bound and a tight lower bound.

Lemma A1: For a serial concatenated channel which consists of two binary symmetric channels (BSCs) with the crossover probabilities being p_1 and p_2, and the mutual information being $I_1=H\left(p_1\right)$ and $I_2=H\left(p_2\right)$, respectively, as shown in Fig. A.1. The mutual information of the serial concatenated channel is

$$I_{\mathrm{SC}} = \mathcal{G}\left(I_1, I_2\right). \tag{A4}$$

Proof: Since the crossover probabilities of the two BSC are p_1 and p_2, respectively, the crossover probability of the serial concatenated channel is

$$p = p_1\left(1-p_2\right)+p_2\left(1-p_1\right) = p_1 + p_2 - 2p_1p_2. \tag{A5}$$

According to the definition of $H(p)$ in (29), the mutual information of the serial concatenated channel is

$$I_{\mathrm{SC}} = 1 - H\left(p\right) = \mathcal{G}\left(I_1, I_2\right). \tag{A6}$$

Theorem A1: Consider a two hop memoryless relay channels, denoted by $S \to R \to T$, the channel in the first hop is a BSC with mutual information I_{hop1} and that in the second hop is a binary input symmetric memoryless output (BISMO) channel with mutual information I_{hop2}. Then the capacity between S and T, denoted by I_{total}, is bounded by

$$I_{\mathrm{hop1}} \cdot I_{\mathrm{hop2}} \le I_{\mathrm{total}} \le \mathcal{G}\left(I_{\mathrm{hop1}}, I_{\mathrm{hop2}}\right). \tag{A7}$$

Proof: As shown in [27], the BISMO channel in the second hop can be decomposed it into a set of BSCs. Define $J \triangleq \left\{j_1, j_2, j_3, j_4, \cdots\right\}$ with $j_n \ge 0$ be the set which consists of the amplitude values of the possible outputs of the second channel. Fig. A.2 provides an example of the BISMO channel decomposition. Define the crossover probability of each BSC in the decomposition of the BISMO channel as

$$q\left(j_n\right) \triangleq \begin{cases} \Pr\left(T = -j_n \middle| R = +1 \text{ and } T = \pm j_n\right) & \text{if } j_n \ne 0 \\ 1/2 & \text{if } j_n = 0 \end{cases}. \tag{A8}$$

The mutual information of each BSC is given by

$$I\left(j_n\right) \triangleq \mathcal{I}\left(R;T \middle| T = \pm j_n\right) = 1 - H\left(q\left(j_n\right)\right). \tag{A9}$$

By averaging over all possible j_n, the mutual information of the BISMO channel in the second hop is given by

$$I_{\mathrm{hop2}} = \underset{j_n \in J}{E}\left\{I\left(j_n\right)\right\}. \tag{A10}$$

By considering the BSC channel in the first hop and the decomposition of the BISMO channel in the second hop together, the total serial concatenated channel can be decomposed into a group of small serial concatenated channels with each being made up of two BSCs. According to Lemma A1, the mutual information of the small serial concatenated channel with the output j_n is given by $\mathcal{G}\left(I_{\text{hop1}}, I\left(j_n\right)\right)$. Averaging it over all possible j_n yields

$$I_{\text{total}} = \underset{j_n \in J}{E}\left\{\mathcal{G}\left(I_{\text{hop1}}, I\left(j_n\right)\right)\right\}. \tag{A11}$$

Since it has been shown in [27] that $\mathcal{G}(a,b)$ is concave for both a or b and $\mathcal{G}(a,b) \geq a \cdot b$, it can found that

$$I_{\text{hop1}} \cdot I_{\text{hop2}} = \underset{j_n \in J}{E}\left\{I_{\text{hop1}} I\left(j_n\right)\right\} \leq I_{\text{total}} \leq \mathcal{G}\left(I_{\text{hop1}}, \underset{j_n \in J}{E}\left\{I\left(j_n\right)\right\}\right) = \mathcal{G}\left(I_{\text{hop1}}, I_{\text{hop2}}\right). \tag{A12}$$

The received signal $y_{3,2}^{\text{Hard}} \triangleq h_{3,2}\text{sign}\left(h_v\left(x_1 \cdot x_2\right) + w_v\right) + w_{3,2}$ can be regarded as the output of a serial concatenated channel. The channel in the first hop whose input is $x_1 \cdot x_2$ and the output is $\text{sign}\left(h_v\left(x_1 \cdot x_2\right) + w_v\right)$ can be regarded as a BSC. The Gaussian channel in the second hop belongs to BISMO channels, it can be obtained directly from *Theorem A1* that

$$C_{\text{BPSK}}^{\text{Hard}}\left(h_v\right) \cdot C_{\text{BPSK}}\left(h_{3,2}\right) \leq \mathcal{I}_{\left(h_{3,2}, h_v\right)}\left(\left(x_1 \cdot x_2\right); y_{3,2}^{\text{Hard}}\right) \leq \mathcal{G}\left(C_{\text{BPSK}}^{\text{Hard}}\left(h_v\right), C_{\text{BPSK}}\left(h_{3,2}\right)\right) \tag{A13}$$

where $C_{\text{BPSK}}^{\text{Hard}}\left(h_v\right)$ and $C_{\text{BPSK}}\left(h_{3,2}\right)$ denote the mutual information of the channels in the two hops, respectively.

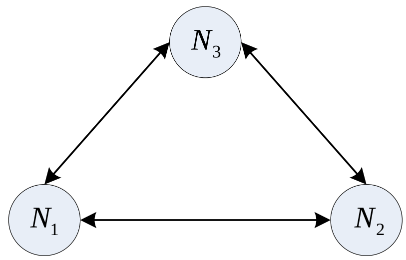

Fig. 1: Two-way relay channels.

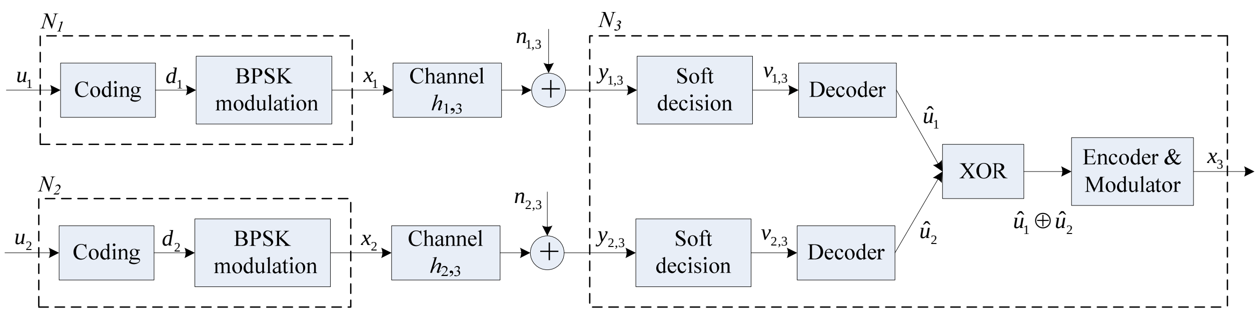

Fig. 2: System diagram of the traditional network coding scheme.

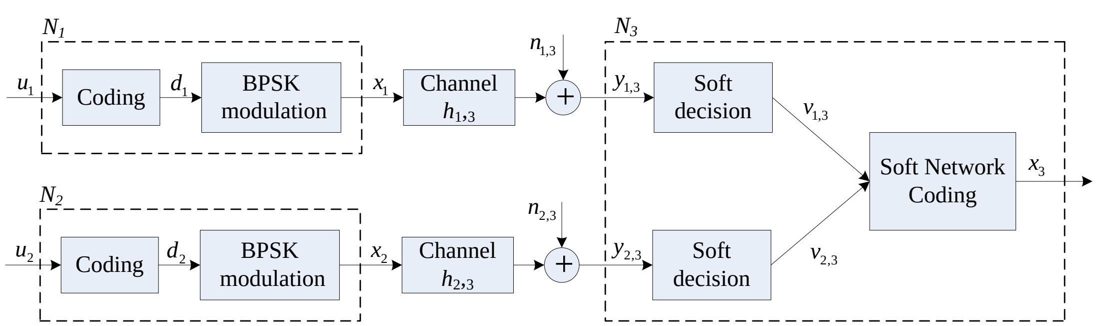

Fig. 3: System diagram of the proposed soft network coding scheme.

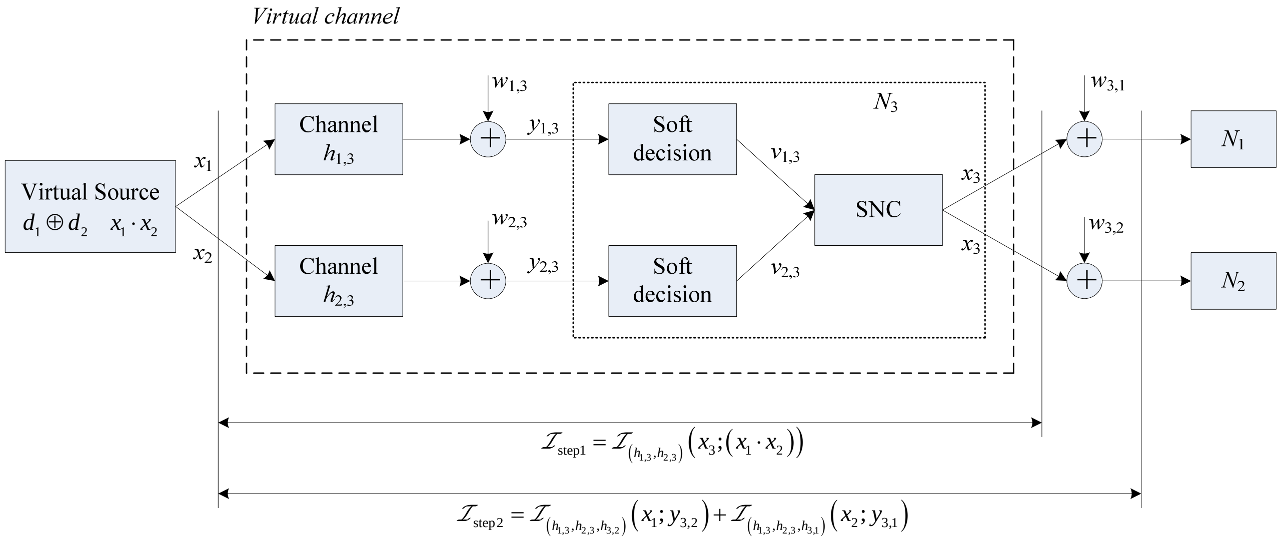

Fig. 4: Mutual information notations in the first and second steps in the analysis for the proposed AF-SoftNC and SBF-SoftNC schemes.

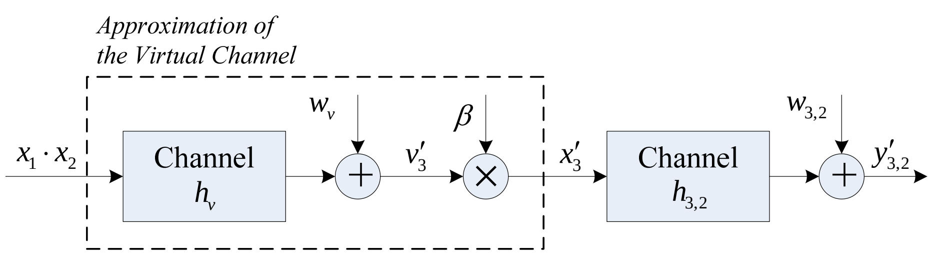

Fig. 5: Approximation of the virtual channel in AF-SoftNC.

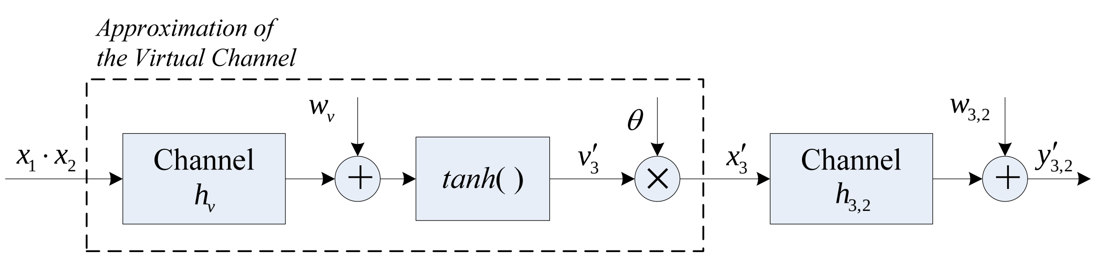

Fig. 6: Approximation of the virtual channel in SBF-SoftNC.

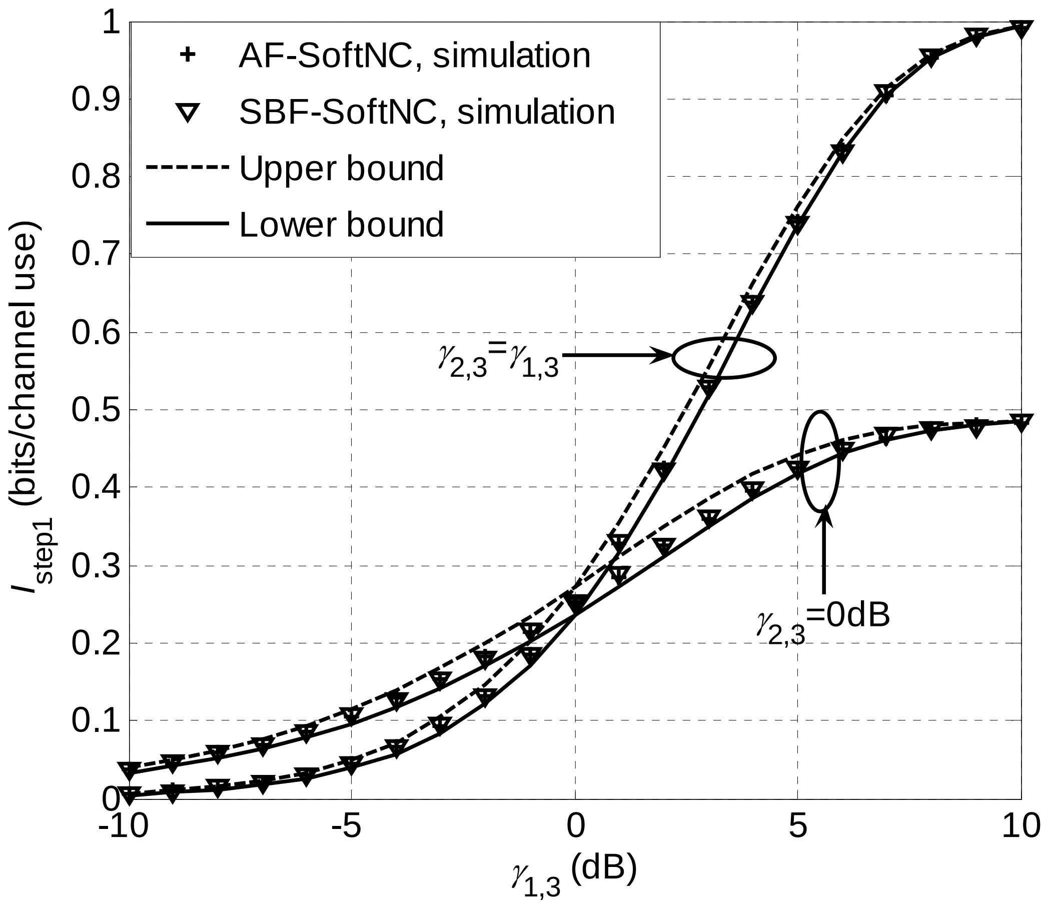

Fig. 7: Bounds of $\mathcal{I}_{\text{step1}}$ for AF-SoftNC and SBF-SoftNC.

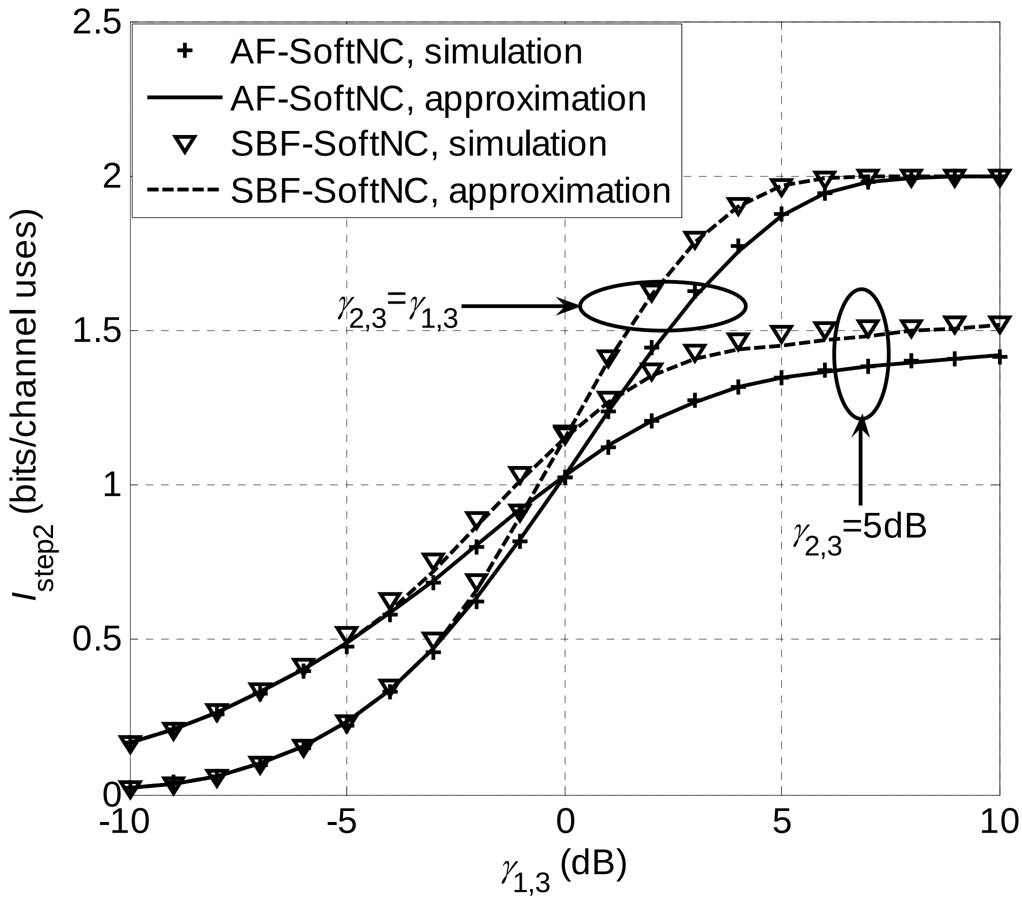

Fig. 8: Approximation of $\mathcal{I}_{\text{step2}}$ for AF-SoftNC and SBF-SoftNC.

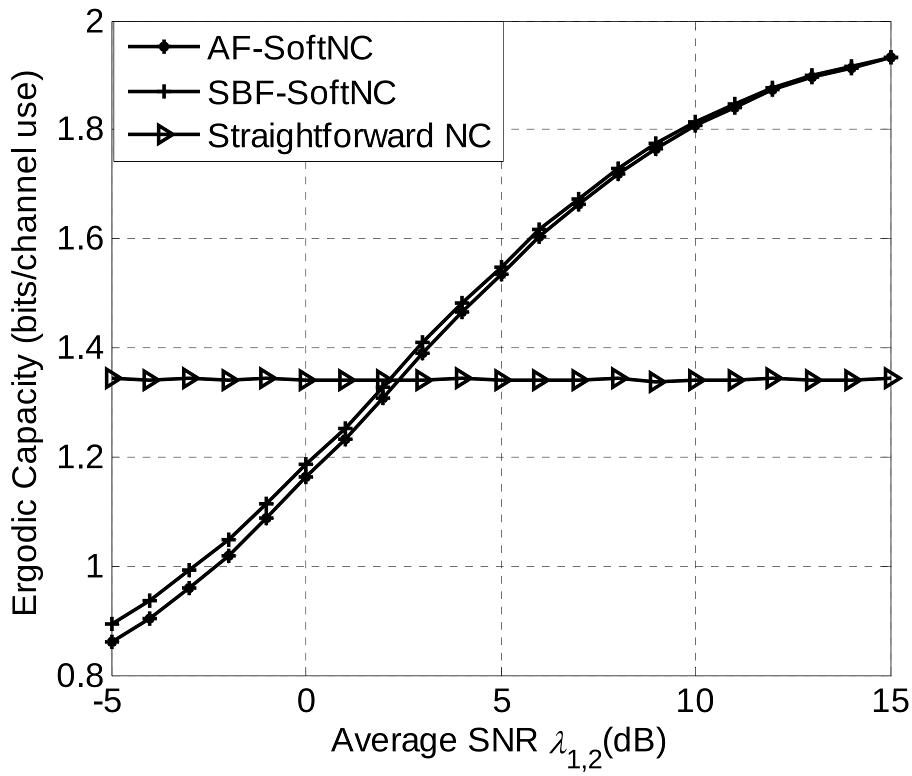

Fig. 9: Ergodic capacity comparison when $\lambda_{1,3} = \lambda_{2,3} = 5\,\text{dB}$ and $\lambda_{1,2}$ varies from -5dB to 15dB.

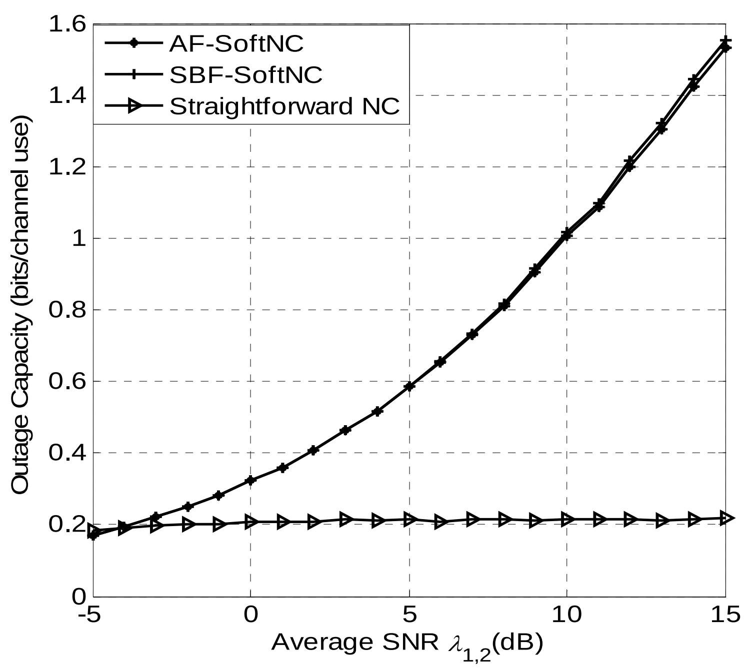

Fig. 10: Outage capacity comparison when $\lambda_{1,3} = \lambda_{2,3} = 5\,\text{dB}$ and $\lambda_{1,2}$ varies from -5dB to 15dB.

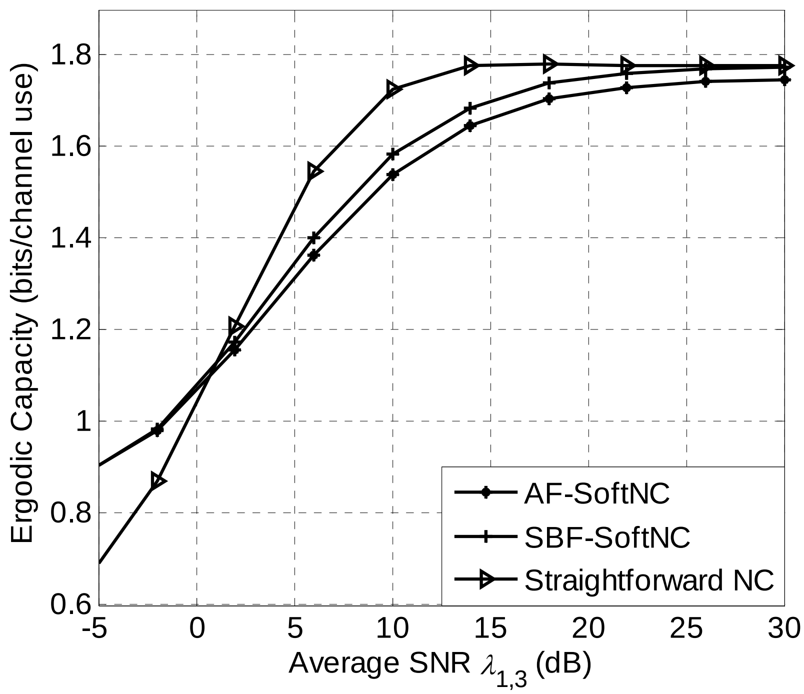

Fig. 11: Ergodic capacity comparison when $\lambda_{1,2} = 0\text{dB}$, $\lambda_{2,3} = 10\,\text{dB}$ and $\lambda_{1,3}$ varies from -10dB to 30dB.

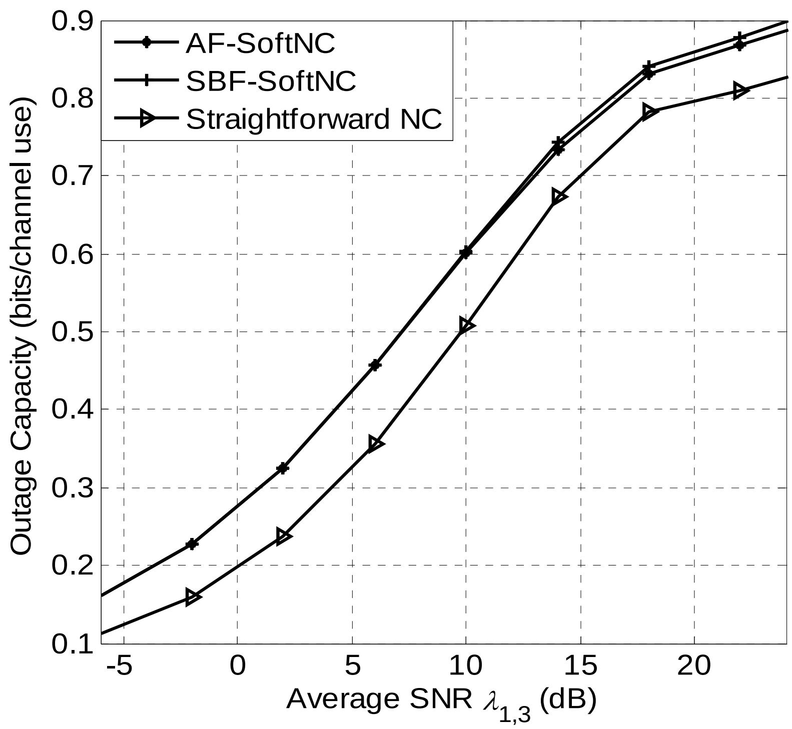

Fig. 12: Outage capacity comparison when $\lambda_{1,2} = 0\text{dB}$, $\lambda_{2,3} = 10\,\text{dB}$ and $\lambda_{1,3}$ varies from -10dB to 30dB.

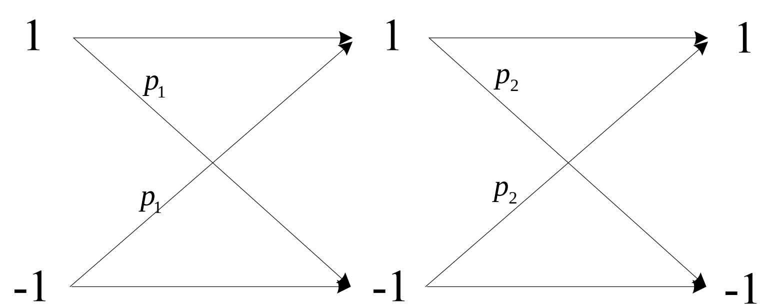

Fig. A1: The serial concatenation of two BSCs.

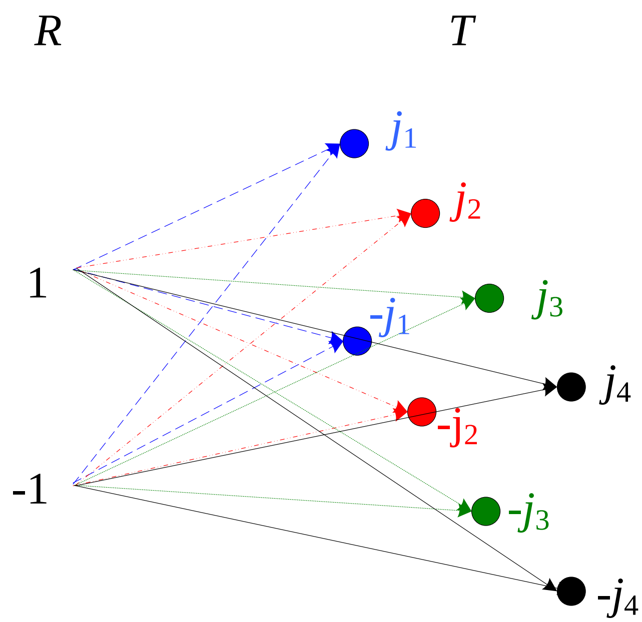

Fig. A2: Decomposition a BISMO channel into a set of BSCs.