Solutions for Written Assignment 1 CSCI 2100A 2016 Spring

Exercise 1.1

4. $\sum_{i=1}^{n} i$

Let $S = \sum_{i=1}^{n} i = 1 + 2 + \cdots + (n-1) + n \quad (1)$

Then $S = n + (n-1) + \cdots + 2 + 1 \quad (2)$

$$(1) + (2) => 2S = \underbrace{(1+n) + (1+n) + \cdots + (1+n) + (1+n)}_{n(1+n)}$$

So $2S = n(1+n), S = \frac{n(n+1)}{2}$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(6) $\sum_{i=1}^{n} ia^i$

Solution: According to the summation formula of geometric progression, $T = \sum_{i=1}^{n} a^i = \frac{a^{n+1}-a}{a-1}$.

Let $S = \sum_{i=1}^{n} ia^i$ $aS = \sum_{i=1}^{n} ia^{(}i+1) = \sum_{i=1}^{n}(i-1)a^i + na^{n+1}$.

$(a-1)S = na^{n+1} - T \Rightarrow S = \frac{1}{a-1}(na^{n+1} - \frac{a^{n+1}-a}{a-1})$.

(12) Is $2^{n+1} = O(2^n)$?

Solution: Yes. $2^{n+1} = 2 \cdot 2^n = O(2^n)$

Exercise 1.3

(3) (in courtesy of Mr Yuen Chin Ki);

1.3 3) $T(n) = 2T(n-1) + n^2$; $T(1) = 1$

Let $T(n) = h(n) + f(n)$,

$f(n) = an^2 + bn + c = 2f(n-1) + n^2$

$an^2 + bn + c = 2a(n-1)^2 + 2b(n-1) + 2c + n^2$

$an^2 + bn + c = 2an^2 - 4an + 2a + 2bn - 2b + 2c + n^2$

$(a - 2a - 1)n^2 + (b + 4a - 2b)n + (c - 2a + 2b - 2c) = 0$

$-a - 1 = 0$; $4a - b = 0$; $-2a + 2b - c = 0$

$a = -1$; $b = -4$; $c = -6$

$\therefore f(n) = -n^2 - 4n - 6$

$h(n) = 2h(n-1)$, characteristic eq: $x^2 = 2x \Rightarrow x = 2 \,/\, x = 0$

i.e. $h(n) = k \cdot 2^n$.

$T(n) = h(n) + f(n)$

$= k \cdot 2^n - n^2 - 4n - 6$

9. Solve $T(1) = 1$, and for all $n \geq 2$ a power of 2, $T(n) = 2T(n/2) + 6n - 1$.

$T(n) = 2T\left(\frac{n}{2}\right) + 6n - 1$

$= 2\left(2T\left(\frac{n}{2^2}\right) + 6 \cdot \frac{n}{2} - 1\right) + 6n - 1$

$= 2^2 T\left(\frac{n}{2^2}\right) + 6n - 2 + 6n - 1$

$= \cdots$

$= nT(1) + (\log_2 n) \cdot (6\text{n}) - (1 + 2 + 2^2 + \cdots + 2^{\log_2 n - 1}$

$= n + 6n \ \cdot \log_2 n - \dfrac{1 - 2^{\log_2 n}}{1 - 2}$

$= n + 6n \cdot \log_2 n + 1 - n$

$= 6n \cdot \log_2 n + 1$

So, $T_n = 6n \cdot \log_2 n + 1$

Exercise 1.4

(1) (in courtesy of Mr Yuen Chin Ki)

1.4 1) $\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$

Base case: When $n = 1$:

$$\frac{1}{2^1} = 1 - \frac{1}{2^1}$$

$\therefore$ The statement holds for $n = 1$.

Inductive step: Assume the statement is true for $n = k$,

When $n = k + 1$:

$$\sum_{i=1}^{k+1} \frac{1}{2^i} = \sum_{i=1}^{k} \frac{1}{2^i} + \frac{1}{2^{k+1}}$$

$$= 1 - \frac{1}{2^k} + \frac{1}{2} \cdot \frac{1}{2^k}$$

$$= 1 - \frac{1}{2} \cdot \frac{1}{2^k} = 1 - \frac{1}{2^{k+1}}$$

$\therefore$ The statement holds for $n = k + 1$.

By Induction, the statement is true for $n \geq 1$.

(3) (in courtesy of Mr Yuen Chin Ki)

3) $\sum_{i=1}^{n}(2i-1)=n^2$

Base case: When $n=1$:

$$(2\cdot 1-1)=(1)^2$$

$\therefore$ The statement holds for $n=1$.

Inductive step: Assume the statement is true for $n=k$,

When $n=k+1$:

$$\sum_{i=1}^{k+1}(2i+1) = \sum_{i=1}^{k}(2i+1) + [2(k+1)+1]$$

$$= k^2+2k+1 = (k+1)^2$$

$\therefore$ The statement holds for $n=k+1$.

By Induction, the statement is true for $n \geq 1$.

(5) Prove $2lg(n!) > nlgn$ by using Induction, where n is a positive integer greater than 2.
Solution: Let $P(n)$ be $lg(n!) > nlgn$, where n is a positive integer.
For $n = 1$, $L.H.S = 2lg(1!) > lg(1) = R.H.S.$ P(1) is true.
Assume P(k) is true, i.e. $2lg(k!) > klgk$, where k is a positive integer
For $n = k + 1$,

$$\begin{aligned}
\text{L.H.S} &= 2lg((k+1)!) \\
&= 2(lg(k!) + lg(k+1)) \\
&> klgk + 2lg(k+1) && \text{(by assumption)} \\
&> (k-1)lg(k+1) + 2lg(k+1) && (k+1 > e > (1+\tfrac{1}{k})^k \Rightarrow k^k > (k+1)^{k-1} \text{ for } k \geq 2) \\
&= (k+1)lg(k+1) \\
&= \text{R.H.S}
\end{aligned}$$

$P(k+1)$ is also true.
Therefore, by M.I., $P(n)$ is true for all positive integer n.
(6) The number generated by the formula $n^2 + n + 17$ is prime for $n \geq 0$, where n is an integer.
Either prove it or disprove it by counterexample.
Solution: No. Let $n = 17$. Then $n^2 + n + 17 = 17 \times (17 + 1 + 1) = 17 \times 19$