Rank aggregation via nuclear norm minimization

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Which is a better list of good DVDs?

Lord of the Rings 3: The Return of …	Lord of the Rings 3: The Return of …
Lord of the Rings 1: The Fellowship	Lord of the Rings 1: The Fellowship
Lord of the Rings 2: The Two Towers	Lord of the Rings 2: The Two Towers
Lost: Season 1	Star Wars V: Empire Strikes Back
Battlestar Galactica: Season 1	Raiders of the Lost Ark
Fullmetal Alchemist	Star Wars IV: A New Hope
Trailer Park Boys: Season 4	Shawshank Redemption
Trailer Park Boys: Season 3	Star Wars VI: Return of the Jedi
Tenchi Muyo!	Lord of the Rings 3: Bonus DVD
Shawshank Redemption	The Godfather

Standard
rank aggregation
(the mean rating)

Nuclear Norm
based rank aggregation
*(**not** matrix completion on the netflix rating matrix)*

Rank Aggregation

Given partial orders on subsets of items, rank aggregation is the problem of finding an overall ordering.

Voting Find the winning candidate

Program committees Find the best papers given reviews

Dining Find the best restaurant in San Diego *(subject to a budget?)*

*Ranking is **really** hard*

Ken Arrow

All rank aggregations involve some measure of compromise

John Kemeny

A good ranking is the "average" ranking under a permutation distance

Dwork, Kumar, Naor, Sivikumar

NP hard to compute Kemeny's ranking

Embody chair
John Cantrell (flickr)

Given a hard problem, what do you do?

Numerically relax!

It'll probably be easier.

Suppose we had scores

Let s_i be the score of the ith movie/song/paper/team to rank

Suppose we can compare the ith to jth:

$$Y_{i,j} = s_i - s_j$$

Then $\mathbf{Y} = \mathbf{s}\mathbf{e}^T - \mathbf{e}\mathbf{s}^T = -\mathbf{Y}^T$ is skew-symmetric, rank 2.

Also works for $Y_{i,j} = s_i / s_j$ with an extra log.

Numerical ranking is intimately intertwined with skew-symmetric matrices

Kemeny and Snell, Mathematical Models in Social Sciences (1978)

Using ratings as comparisons

Ratings induce various skew-symmetric matrices.

$$Y_{i,j} = \frac{\sum_u R_{u,i} - R_{u,j}}{|\{u \mid R_{u,i} \text{ and } R_{u,j} \text{ exist}\}|}$$ *Arithmetic Mean*

$$Y_{i,j} = \log \frac{\Pr_u(R_{u,i} \geq R_{u,j})}{\Pr_u(R_{u,i} \leq R_{u,j})}$$ *Log-odds*

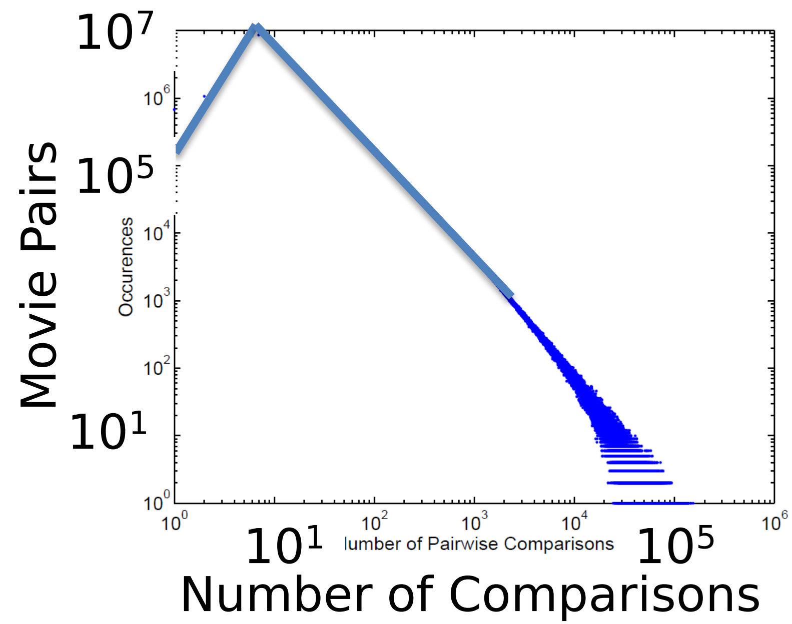
David 1988 – The Method of Paired Comparisons

Extracting the scores

Given $\boldsymbol{Y}$ with all entries, then
$\mathbf{s} = \frac{1}{n}\mathbf{Y}\mathbf{e}$ is the *Borda count*, the least-squares solution to $\mathbf{s}$

How many $Y_{i,j}$ do we have?
Most.

Do we *trust* all $Y_{i,j}$?
Not really.



Netflix data 17k movies, 500k users, 100M ratings– 99.17% filled

Only partial info? ***Complete it!***

Let $\hat{Y}_{i,j}$ be known for $(i,j) \in \Omega$ *We trust these scores*.

Goal Find the simplest skew-symmetric matrix that matches the data $\hat{Y}_{i,j}$

noiseless

$$\begin{array}{ll} \text{minimize} & \text{rank}(\boldsymbol{Y}) \\ \text{subject to} & \boldsymbol{Y} = -\boldsymbol{Y}^T \\ & Y_{i,j} = \hat{Y}_{i,j} \text{ for all } (i,j) \in \Omega \end{array}$$

noisy

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in\Omega} (Y_{i,j} - \hat{Y}_{i,j})^2 + \lambda \text{rank}(\mathbf{Y}) \\ \text{subject to} & \mathbf{Y} = -\mathbf{Y}^T \end{array}$$

Both of these are NP-hard too.

Solution **Go Nuclear**

From a French nuclear test in 1970, image from http://picdit.wordpress.com/2008/07/21/8-insane-nuclear-explosions/

*The **nuclear** norm*

The analog of the 1-norm or ℓ_1 for matrices.

For vectors

minimize $\|\mathbf{Ax} - \mathbf{b}\|^2 + \lambda \operatorname{nnz}(\mathbf{x})$

is NP-hard while

minimize $\|\mathbf{Ax} - \mathbf{b}\|^2 + \lambda \|\mathbf{x}\|_1$

is convex and gives the same answer "under appropriate circumstances"

For matrices

Let $\mathbf{A} = \mathbf{U\Sigma V}^T$ be the SVD.

$\operatorname{nnz}(\mathbf{x}) \sim \operatorname{rank}(\mathbf{A}) = \operatorname{nnz}(\mathbf{\Sigma})$

$\|\mathbf{x}\|_1 \sim \|\mathbf{A}\|_* = \|\operatorname{diag}(\mathbf{\Sigma})\|_1 = \sum \sigma_i$

$\|\mathbf{Y}\|_*$ best convex under-estimator of rank on unit ball.

Only partial info? ***Complete it!***

Let $\hat{Y}_{i,j}$ be known for $(i,j) \in \Omega$ *We trust these scores.*

Goal Find the simplest skew-symmetric matrix that matches the data $\hat{Y}_{i,j}$

NP hard

$$\begin{array}{ll} \text{minimize} & \text{rank}(\boldsymbol{Y}) \\ \text{subject to} & \boldsymbol{Y} = -\boldsymbol{Y}^T \\ & Y_{i,j} = \hat{Y}_{i,j} \text{ for all } (i,j) \in \Omega \end{array}$$

Heuristic

Convex

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{Y}\|_* \\ \text{subject to} & \boldsymbol{Y} = -\boldsymbol{Y}^T \\ & Y_{i,j} = \hat{Y}_{i,j} \text{ for all } (i,j) \in \Omega \end{array}$$

Solving the nuclear norm problem

Use a LASSO formulation

$\mathbf{b} = \text{vec}(\hat{Y}_{i,j})$

$$\begin{array}{ll} \text{minimize} & \|\Omega(\boldsymbol{Y}) - \mathbf{b}\| \\ \text{subject to} & \|\boldsymbol{Y}\|_* \le 2 \\ & \boldsymbol{Y} = -\boldsymbol{Y}^T \end{array}$$

Jain et al. propose SVP for this problem without $\boldsymbol{Y} = -\boldsymbol{Y}^T$

1. $\boldsymbol{Y}_0 = 0, t = 0$
2. REPEAT
3. $\boldsymbol{U}_t \boldsymbol{\Sigma}_t \boldsymbol{V}_t^T$ = rank-k SVD of $\Omega(\boldsymbol{Y}_t) - \eta(\Omega(\boldsymbol{Y}_t) - \mathbf{b})$
4. $\boldsymbol{Y}_{t+1} = \boldsymbol{U}_t \boldsymbol{\Sigma}_t \boldsymbol{V}_t^T$
5. $t = t + 1$
6. UNTIL $\|(\Omega(\boldsymbol{Y}_t) - \mathbf{b})\| < \varepsilon$

Jain et al. NIPS 2010

Skew-symmetric SVDs

Let $\boldsymbol{A} = -\boldsymbol{A}^T$ be an $n \times n$ skew-symmetric matrix with eigenvalues $i\lambda_1, -i\lambda_1, i\lambda_2, -i\lambda_2, \ldots, i\lambda_j, -i\lambda_j$, where $\lambda_i > 0, \lambda_i \geq \lambda_{i+1}$ and $j = \lfloor n/2 \rfloor$. Then the SVD of $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \boldsymbol{U} \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_1 & & & & & \\ & & \lambda_2 & & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_j & \\ & & & & & & \lambda_j \end{bmatrix} \boldsymbol{V}^T$$

for $\boldsymbol{U}$ and $\boldsymbol{V}$ given in the proof.

Proof Use the Murnaghan-Wintner form and the SVD of a 2x2 skew-symmetric block

This means that SVP will give us the skew-symmetric constraint "for free"

Exact recovery results

David Gross showed how to recover Hermitian matrices.
i.e. the conditions under which we get the exact **s**

Note that ***ιY*** is Hermitian. Thus our new result!

THEOREM 5. *Let* $\mathbf{s}$ *be centered, i.e.,* $\mathbf{s}^T\mathbf{e} = 0$. *Let* $\boldsymbol{Y} = \mathbf{s}\mathbf{e}^T - \mathbf{e}\mathbf{s}^T$ *where* $\theta = \max_i s_i^2/(\mathbf{s}^T\mathbf{s})$ *and* $\rho = ((\max_i s_i) - (\min_i s_i))/\|\mathbf{s}\|$. *Also, let* $\Omega \subset \mathcal{H}$ *be a random set of elements with size* $|\Omega| \geq O(2n\nu(1+\beta)(\log n)^2)$ *where* $\nu = \max((n\theta + 1)/4, n\rho^2)$. *Then the solution of*

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{X}\|_* \\ \text{subject to} & \operatorname{trace}(\boldsymbol{X}^*\boldsymbol{W}_i) = \operatorname{trace}((\imath\boldsymbol{Y})^*\boldsymbol{W}_i), \quad \boldsymbol{W}_i \in \Omega \end{array}$$

is equal to $\imath\boldsymbol{Y}$ *with probability at least* $1 - n^{-\beta}$.

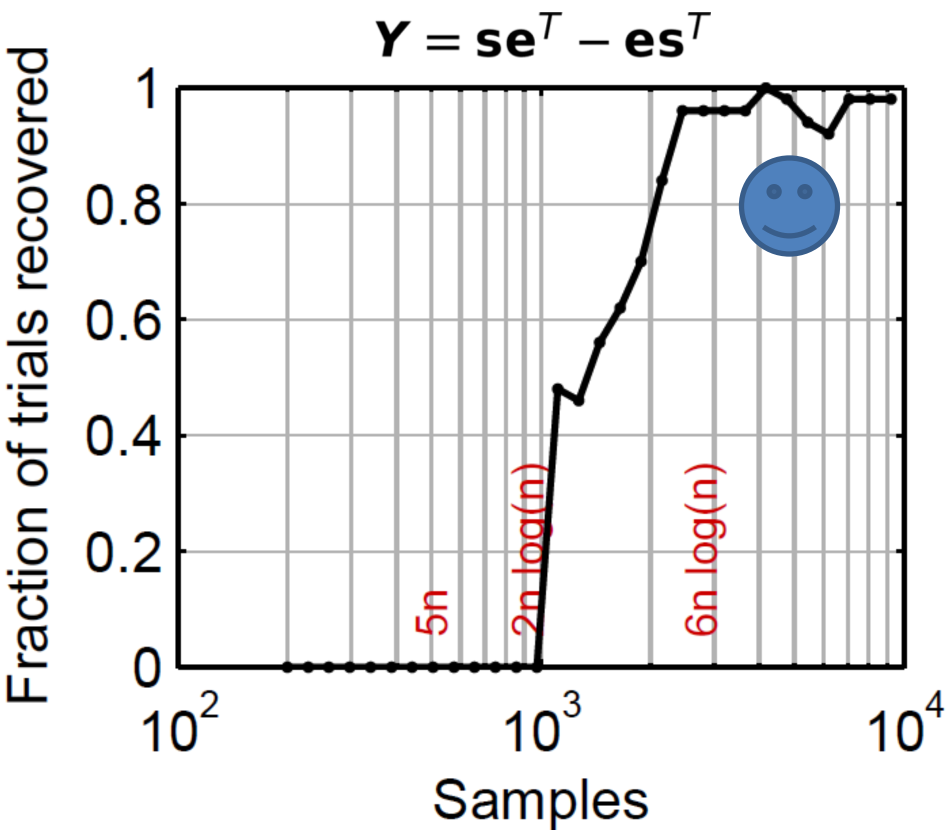
"$n \log(n)$"

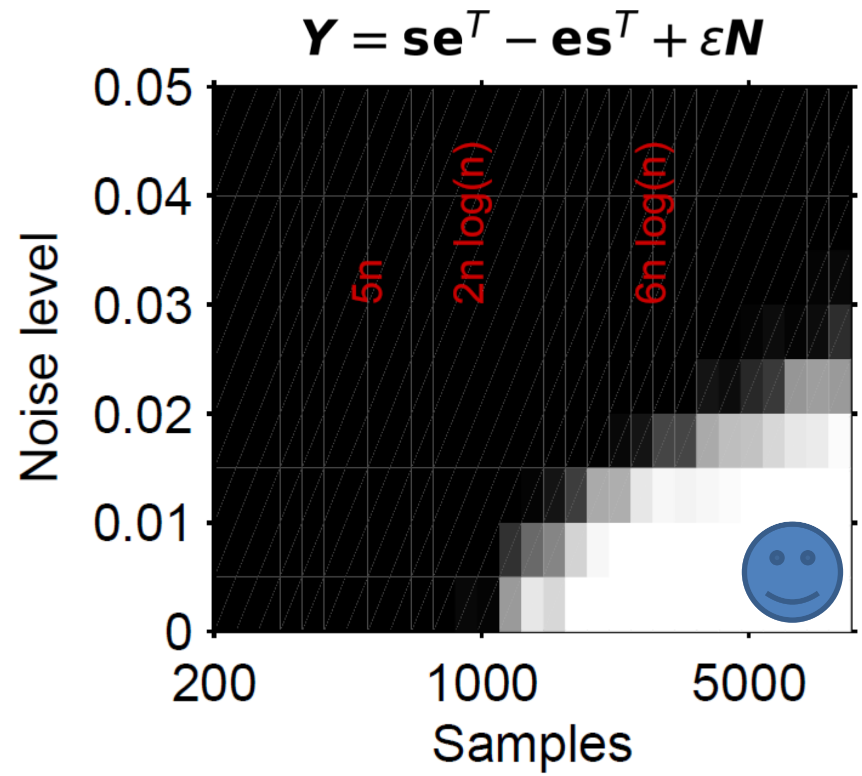
Gross arXiv 2010.

Recovery Discussion and Experiments

Confession If $\mathbf{Y} = \mathbf{s}\mathbf{e}^T - \mathbf{e}\mathbf{s}^T$, then just look at differences from a connected set. Constants? Not very good.

"$n \log(n)$" **Intuition for *the truth*.**





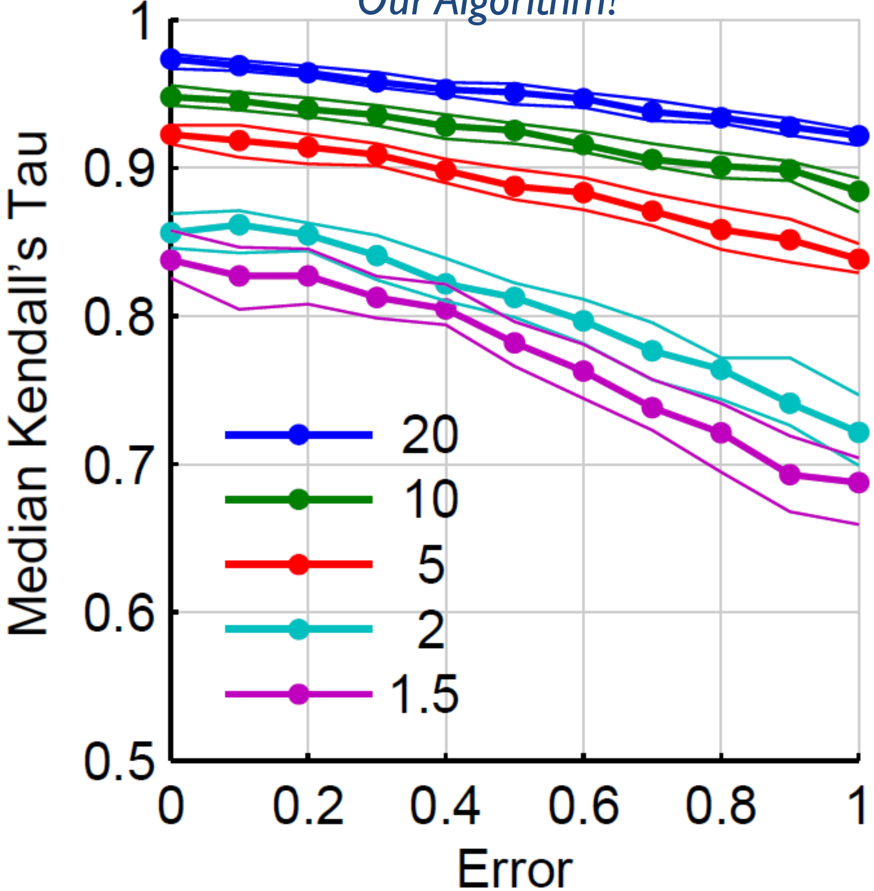
The Ranking Algorithm

0. INPUT $\boldsymbol{R}$ (ratings data) and c (for trust on comparisons)
1. Compute $\boldsymbol{Y}$ from $\boldsymbol{R}$
2. Discard entries with fewer than c comparisons
3. Set $\Omega, \mathbf{b}$ to be indices and values of what's left
4. $\boldsymbol{U}, \boldsymbol{S}, \boldsymbol{V}^T = \text{SVP}(\Omega, \mathbf{b}, 2)$
5. OUTPUT $\mathbf{s} = (1/n)\boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T\mathbf{e}$

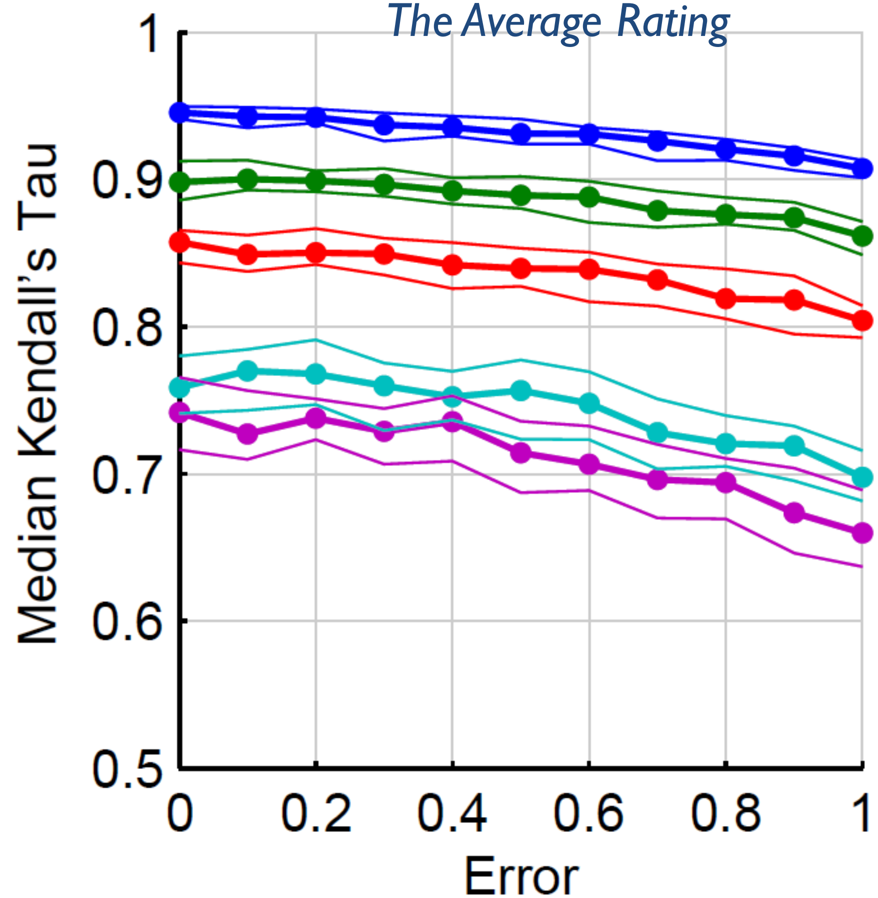
Synthetic Results

Construct an Item Response Theory model. Vary number of ratings per user and a noise/error level

Our Algorithm!



The Average Rating



Conclusions and Future Work

"aggregate, then complete"

Rank aggregation with the nuclear norm is

principled

easy to compute

The results are much better than simple approaches.

1. Compare against others
2. Noisy recovery! More realistic sampling.
3. Skew-symmetric Lanczos based SVD?

Google nuclear ranking gleich

https://dgleich.com/projects/skew-nuclear