Some Recent Advances in the Theory of Low-rank Modeling

Emmanuel Candès

Machine Learning Summer School (MLSS 2001)
Carcans Maubuisson, September 2011

Objective

- Explosion of research on theory of low-rank modeling
- Our goal is to discuss some recent works
    - Some of it is ours
    - Some of it is not

Agenda

1. Matrix completion
2. Robust principal component analysis

Matrix Completion

The Netflix problem

- Netflix database
    - About half a million users
    - About 18,000 movies
- People rate movies
- Sparsely sampled entries

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\begin{array}{cc}
 & \text{Movies} \\
\text{Users} & \begin{bmatrix}
\times & & & \times & \\
 & \times & \times & & \\
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\times & & & & \\
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\end{bmatrix}
\end{array}
$$

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$$

Challenge

Complete the "Netflix matrix"

Many such problems → collaborative filtering, partially filled out surveys...

Global positioning from local distances

- Points $\{x_j\}_{1 \le j \le n} \in \mathbb{R}^d$
- Partial information about distances

$$L_{ij} = \|x_i - x_j\|^2$$

Example (Singer, Biswas et al.)

- Low-powered wirelessly networked sensors
- Each sensor can construct a distance estimate from nearest neighbor

$$\begin{array}{cc} & j \\ i & \begin{bmatrix} \times & & & \times & \\ & \times & \times & & \\ \times & & \times & & \\ & \times & & & \times \\ \times & & & & \\ & \times & \times & & \end{bmatrix} \end{array}$$

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Problem

Locate the sensors

Other problems of this kind

- Linear system identification (Vandenberghe et al.)
- Quantum-state tomography (Gross et al.)
- Partially observed covariance matrix (Vaidyanathan et al.)
- Low-rank matrix completion in machine learning (Srebro et al. Vert et al.)
- Structure-from-motion problem in computer vision (Tomasi et al.)
- ...

Matrix completion

- Matrix $L \in \mathbb{R}^{n_1 \times n_2}$
- Observe subset of entries
- Can we guess the missing entries?

$$\begin{bmatrix} \times & ? & ? & ? & \times & ? \\ ? & ? & \times & \times & ? & ? \\ \times & ? & ? & \times & ? & ? \\ ? & ? & \times & ? & ? & \times \\ \times & ? & ? & ? & ? & ? \\ ? & ? & \times & \times & ? & ? \end{bmatrix}$$

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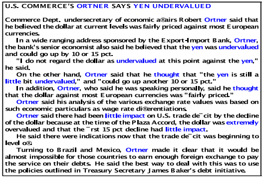
Everybody would agree this looks impossible

Massive high-dimensional data

Engineering/scientific applications: unknown matrix has often (approx.) low rank

Images

Videos



U.S. COMMERCE'S ORTNER SAYS YEN UNDERVALUED

Commerce Dept. undersecretary of economic a®airs Robert Ortner said that he believed the dollar at current levels was fairly priced against most European currencies.

In a wide ranging address sponsored by the Export-Import Bank, Ortner, the bank's senior economist also said he believed that the yen was undervalued and could go up by 10 or 15 pct.

"I do not regard the dollar as undervalued at this point against the yen," he said.

On the other hand, Ortner said that he thought that "the yen is still a little bit undervalued," and "could go up another 10 or 15 pct."

In addition, Ortner, who said he was speaking personally, said he thought that the dollar against most European currencies was "fairly priced."

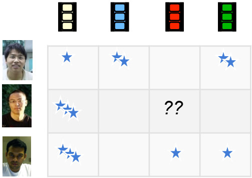
Ortner said his analysis of the various exchange rate values was based on such economic particulars as wage rate di®erentiations.

Ortner said there had been little impact on U.S. trade de¯cit by the decline of the dollar because at the time of the Plaza Accord, the dollar was extremely overvalued and that the ¯rst 15 pct decline had little impact.

He said there were indications now that the trade de¯cit was beginning to level o®.

Turning to Brazil and Mexico, Ortner made it clear that it would be almost impossible for those countries to earn enough foreign exchange to pay the service on their debts. He said the best way to deal with this was to use the policies outlined in Treasury Secretary James Baker's debt initiative.

Text



Web data

High-dimensionality
but often
low-dimensional structure

Low-rank matrix completion?

Engineering/scientific applications: unknown matrix has often (approx.) low rank

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1. Netflix matrix
2. Sensor-net matrix: $\|x_i - x_j\|^2$, $\{x_i\} \in \mathbb{R}^d$
    - rank 2 if $d = 2$
    - rank 3 if $d = 3$
    - ...
3. Many others (e.g. quantum-state tomography, computer vision, system id, ...)

COMMITTEE ON APPLIED & THEORETICAL ·STATISTICS·

Committee on National Statistics
THE NATIONAL ACADEMIES

Announcing a *Joint Seminar* of the **Committee on Applied and Theoretical Statistics** and the **Committee on National Statistics** of *The National Academies*...

THE STORY OF THE Netflix Prize

Friday, November 4, 2011 • 3:00–5:00 pm

Reception to Follow

Keck Center of the National Academies, Room 100
500 Fifth Street NW
Washington, DC 20001

Robert Bell
AT&T Labs Research

Lester Mackey
University of California, Berkeley

Emmanuel Candes
Stanford University

Just over five years ago, Netflix released more than 100 million movie ratings as part of a data analysis contest to improve methods for recommending movies to customers based on ratings they had provided for previously rented movies. A prize of $1 million was offered for a "recommender" algorithm that outperformed the existing Netflix system Cinematch℠ by at least 10% in terms of root mean squared prediction error. In a textbook example of "crowdsourcing," more than 20,000 teams from over 150 countries submitted algorithms. By August 2009, after almost three years of effort, two teams, BellKor's Pragmatic Chaos and The Ensemble, had surpassed the 10% goal in a finish worthy of its own movie.

Bob Bell (*BellKor's Pragmatic Chaos*) and Lester Mackey (*The Ensemble*) will describe the overall set-up of the competition, the challenges it posed, the main ideas underlying their recommender algorithms, and the interaction among the leading competitors. Emmanuel Candes will then discuss the research avenues stimulated by the various algorithms developed in this competition, some of the resulting advances, and some difficult problems that remain.

— Open to the Public • Please RSVP! —

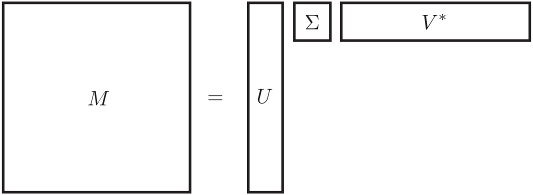
*For planning and building check-in purposes, please RSVP by **October 31** to Agnes Gaskin at agaskin@nas.edu or (202) 334-3096.*

Low-rank matrix completion?

L: $n_1 \times n_2$ matrix of <u>rank r</u>

$$\begin{bmatrix} \times & ? & ? & ? & \times & ? \\ ? & ? & \times & \times & ? & ? \\ \times & ? & ? & \times & ? & ? \\ ? & ? & \times & ? & ? & \times \\ \times & ? & ? & ? & ? & ? \\ ? & ? & \times & \times & ? & ? \end{bmatrix}$$

- Singular value decomposition: $L = U\Sigma V^*$



- L depends upon $(n_1 + n_2 - r)r$ degrees of freedom $<$ ambient dimension

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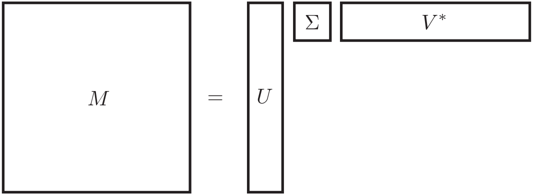

- L depends upon $(n_1 + n_2 - r)r$ degrees of freedom $<$ ambient dimension

Do we need to see all the entries to recover L?

Which entries do we get to see?

Rank-1 matrix $L = xy^*$

$$L_{ij} = x_i y_j$$

$$\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ & & & & & \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix}$$

If single row (or column) is not sampled $\rightarrow$ recovery is not possible

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If single row (or column) is not sampled $\rightarrow$ recovery is not possible

What happens for almost all sampling sets?

m entries selected uniformly at random $\rightarrow \Omega_{\text{obs}}$

Which matrices can we complete?

$$L = e_1 e_n^* = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Cannot be recovered from a small set of entries

Which matrices can we complete?

$$L = \begin{bmatrix} * & * & 0 & \cdots & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

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Cannot be recovered from a small set of entries

Intuition: column and row spaces cannot be aligned with basis vectors

Coherence

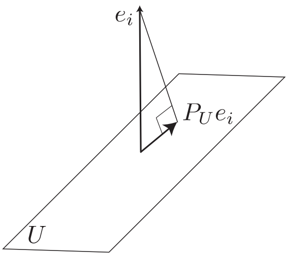
$$L \in \mathbb{R}^{n\times n} = U\Sigma V^* \qquad r = \text{rank}(L)$$

Coherence parameter $\mu \geq 1$ (C. and Recht '08): for all $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)$

$$\|U^* e_i\|^2 \leq \frac{\mu r}{n} \quad \|V^* e_i\|^2 \leq \frac{\mu r}{n}$$

and

$$|UV^*|_{ij}^2 \leq \frac{\mu r}{n^2}$$



Roughly: small value of $\mu \to$ sing. vectors not sparse

Condition holds if $|U_{ij}|^2 \vee |V_{ij}|^2 \leq \mu/n$

Random plane of dimension $r \geq \log n$

$$\max_i \|U^* e_i\|^2 \leq O(1) r/n$$

What is information theoretically possible?

C. and Tao (09)

Roughly, no method whatsoever can succeed

$$m \lesssim \mu \times nr \times \log n \approx \text{df} \times \mu \log n$$

For rectangular matrices $n = \text{max dim}$

- Fundamental role played by coherence parameter
- With $\mu = O(1)$ (incoherence), need $m \gtrsim nr \log n$

Recovery algorithm

Hope: only one low-rank matrix consistent with the sampled entries

Recovery by minimum complexity

$$\begin{array}{ll} \text{minimize} & \text{rank}(\hat{L}) \\ \text{subject to} & \hat{L}_{ij} = L_{ij} \quad (i,j) \in \Omega_{\text{obs}} \end{array}$$

NP-hard: not feasible for $n > 10$!

Recovery algorithm

Hope: only one low-rank matrix consistent with the sampled entries

Recovery by nuclear-norm minimization (SDP)

$$\begin{array}{ll} \text{minimize} & \|\hat{L}\|_* = \sum_{i=1}^n \sigma_i(\hat{L}) \\ \text{subject to} & \hat{L}_{ij} = L_{ij} \quad (i,j) \in \Omega_{\text{obs}} \end{array}$$

- Convex relaxation of the rank minimization program
- Ball $\{X : \|X\|_* \le 1\}$: convex hull of rank-1 matrices obeying $\|xy^*\| \le 1$

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Trace norm heuristics

- Mesbahi & Papavassilopoulos '97
- Beck & D'Andrea '98
- Fazel '02

Near-optimal matrix completion

$$\begin{array}{ll} \text{minimize} & \|\hat{L}\|_* \\ \text{subject to} & \hat{L}_{ij} = L_{ij} \ \ (i,j) \in \Omega_{\text{obs}} \end{array}$$

$$\begin{bmatrix} \times & ? & ? & ? & \times & ? \\ ? & ? & \times & \times & ? & ? \\ \times & ? & ? & \times & ? & ? \\ ? & ? & \times & ? & ? & \times \\ \times & ? & ? & ? & ? & ? \\ ? & ? & \times & \times & ? & ? \end{bmatrix}$$

Theorem (C. and Tao '09 improving C. and Recht '08)

- *rank*$(L) = r$
- Ω_{obs} *random set of size* m

Solution to SDP is exact with probability at least $1 - n^{-10}$ *if*

$$m \gtrsim \mu\, nr \log^a n \quad a \leq 6 \text{ (sometimes 2)}$$

Gross' near-optimal improvement

$$m \gtrsim \mu\, nr \log^2 n$$

Related work

- Related results
  - Recht Parrilo Fazel '07
  - Keshavan, Oh and Montanari '09
- Earlier result [C. and Recht '08]:

$$m \gtrsim \mu n^{6/5} r \log n$$

- Other contributions
  - Cai, C. and Shen '08
  - Mazumder, Hastie and Tibshirani '09
  - Ma and Goldfarb '09
  - ...

Geometry

$$\begin{array}{ll} \text{minimize} & \|\hat{L}\|_* \\ \text{subject to} & \hat{L}_{ij} = L_{ij} \quad (i,j) \in \Omega \end{array}$$

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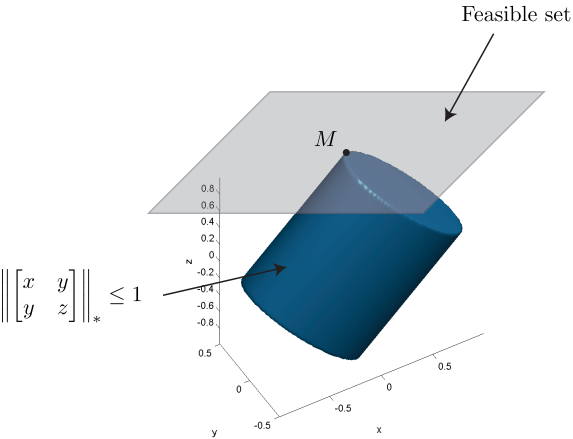

General formulation

- $A_1, \ldots, A_N$ (orthonormal) basis of $\mathbb{R}^{n \times n}$ $(N = n^2)$
- $\Omega \subset \{1, \ldots, N\}$

$$\begin{array}{ll} \text{minimize} & \|X\|_* \\ \text{subject to} & \langle A_k, X \rangle = \langle A_k, L \rangle \quad k \in \Omega \end{array}$$

If incoherence between sensing matrices $\{A_k\}$ and col. + row space

everything should work...

Example: C. and Recht '08

- Two orthonormal bases $F = [f_1, \ldots, f_n]$, $G = [g_1, \ldots, g_n]$
- Orthobasis of $n \times n$ matrices: $\{f_i g_j^*\}_{1 \le i,j \le n}$

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Succeeds if col. (resp. row) space of L incoherent with $\{f_i\}$ (resp. $\{g_i\}$)

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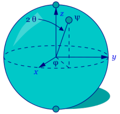
Why? Because $f_i^* X g_j = e_i^* (F^* X G) e_j$

Quantum-state tomography



- k spin-$1/2$ system in an *unknown* quantum state $L \in \mathbb{C}^{n \times n}$ (density matrix)

$$n = 2^k, \quad \text{trace}(L) = 1, \quad L \succcurlyeq 0$$

- Quantum measurements (data)

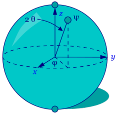
$$\mathbb{E}[\text{measurement with observable } A_j] = \langle A_j, L \rangle = \text{trace}(A_j^* L)$$

e.g. $\{A_j\}$: tensor Pauli matrices

Q? Can we reduce # measurements by using the structure of special classes of quantum states?

- pure state $\rightarrow$ rank$(L) = 1$
- interesting mixed states $\rightarrow$ (approx) low rank

Quantum-state tomography



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A. Yes. Sample in proportion to the rank of the quantum state (Gross '09)

General statement

$A_1, \ldots, A_{n^2}$ (orthonormal) basis of $\mathbb{R}^{n \times n}$ and observe ($L = U\Sigma V^*$)

$$y_k = \langle A_k, L \rangle \quad k \in \Omega$$

Ω random set of size m

- Coherence assumption

$$\max_k \|P_U A_k\|_F^2 \leq \mu r/n \qquad \max_k \|A_k P_V\|_F^2 \leq \mu r/n$$

- At least one of the two conditions

$$\max_k \|A_k\|^2 \leq \mu/n$$
$$\max_k |\langle A_k, UV^* \rangle|^2 \leq \mu r/n^2$$

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Theorem (Gross '09)

Min nuclear-norm solution is exact with high prob. provided

$$m \gtrsim \mu \times nr \times \log^2 n$$

Robust PCA

Matrix completion from noisy entries

$$Y_{ij} = L_{ij} + Z_{ij}, \quad (i,j) \in \Omega_{\text{obs}} \quad Z_{ij} \text{ iid } \mathcal{N}(0, \sigma^2)$$

Recovery by SDP with relaxed constraints

minimize	$\|\hat{L}\|_*$
subject to	$\sum_{ij \in \Omega_{\text{obs}}} (\hat{L}_{ij} - Y_{ij})^2 \leq (1+\epsilon) n \sigma^2$

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Theorem (C. and Plan, '09)

Same assumptions as before. With very high prob.

$$n^{-2} \|\hat{L} - L\|_F^2 \lesssim n\sigma^2$$

When exact recovery occurs, noisy variant is stable

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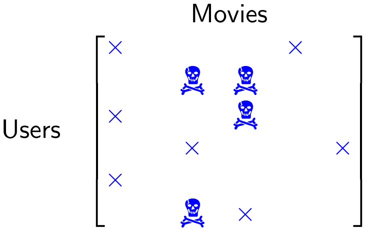
Some other works

- Koltchinskii, Lounici & Tsybakov ('10)
- Bunea, She & Wegkamp ('10)
- Negahban & Wainwright ('10)
- Rohde & Tsybakov ('10)

Gross errors



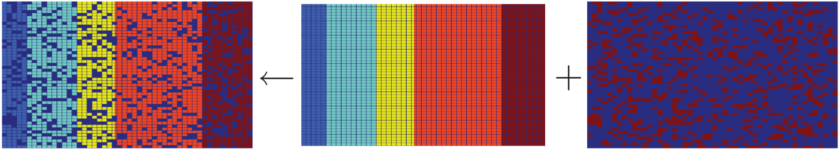
Observe corrupted samples from $L + E$

- L low-rank matrix
- E entries that have been tampered with – impulsive noise

Goal

Recover L: make approach robust vis a vis gross errors

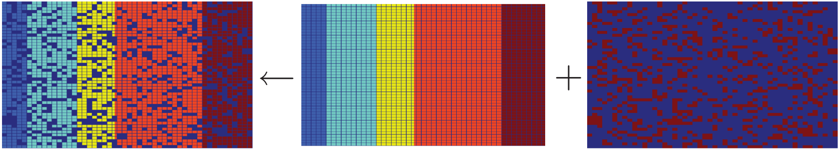
The separation problem



$$M = L + E$$

- M: data matrix (observed)
- L: low-rank (unobserved)
- E: sparse (unobserved)

The separation problem



$$M = L + E$$

- M: data matrix (observed)
- L: low-rank (unobserved)
- E: sparse (unobserved)

Problem: can we recover L and E accurately?

Again, seems impossible

Classical PCA

$$M = L + N$$

- L: low-rank (unobserved)
- N: (small) perturbation

Dimensionality reduction (Schmidt 1907, Hotelling 1933)

$$\begin{array}{ll} \text{minimize} & \|M - \hat{L}\| \\ \text{subject to} & \operatorname{rank}(\hat{L}) \le k \end{array}$$

Solution given by truncated SVD

$$M = U\Sigma V^* = \sum_i \sigma_i u_i v_i^* \quad \Rightarrow \quad \hat{L} = \sum_{i \le k} \sigma_i u_i v_i^*$$

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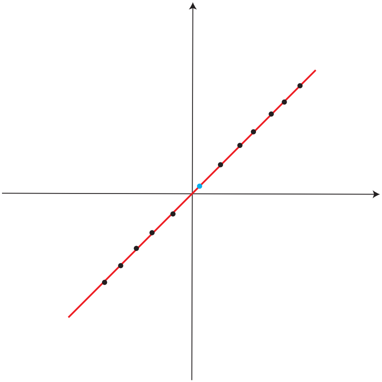
Fundamental statistical tool: enormous impact

PCA and corruptions/outliers

PCA: very sensitive to outliers



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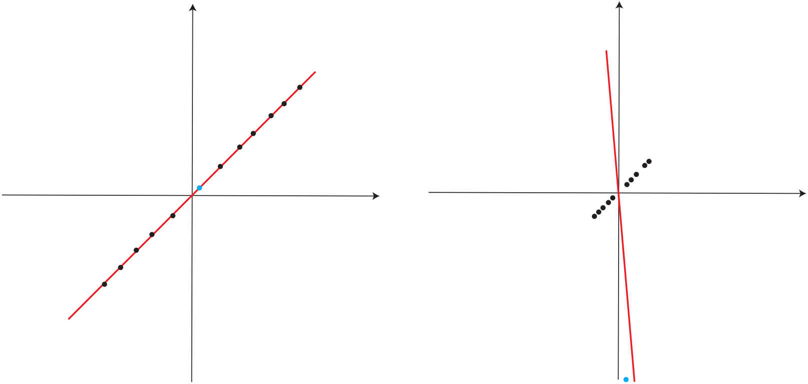


Breaks down with one (badly) corrupted data point

Robust PCA

Gross errors frequently occur in many applications

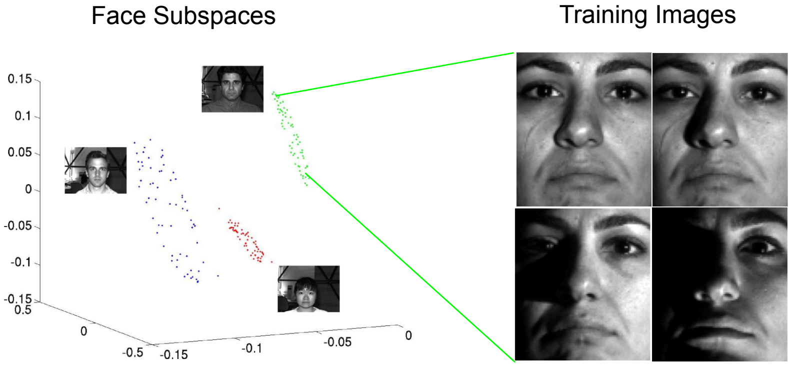
- Image processing
- Web data analysis
- Bioinformatics
- ...

- Occlusions
- Malicious tampering
- Sensor failures
- ...

Important to make PCA robust

- Influence function techniques: Huber; De La Torre and Black
- Multivariate trimming: Gnanadesikan and Kettenring
- Alternating minimization: Ke and Kanade
- Random sampling techniques: Fischler and Bolles
- ...

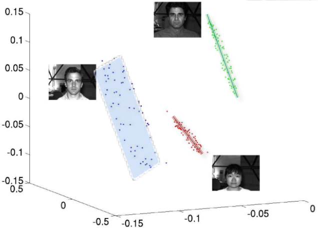
Example: Face recognition under varying illuminations

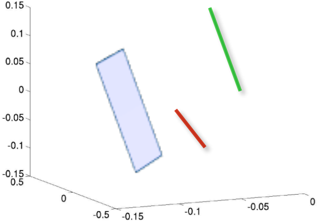


Images of same face under varying illuminations $\sim$ 9D harmonic plane (Basri and Jacobs, 03)

Occlusions and other corruptions in computer vision

?





Real data are corrupted, have missing blocks → classical methods break down

How do we develop provably correct and efficient algorithms for recovery of low-dimensional linear structure from non-ideal observations?

When does separation make sense?

What if $M = L + E$ is both low-rank and sparse?

$$M = e_1 e_n^* = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

When does separation make sense?

What if $M = L + E$ is both low-rank and sparse?

$$M = e_1 e_n^* = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Low-rank component cannot be sparse

Will assume $L \in \mathbb{R}^{n \times n}$ obeys previous incoherence condition

"sing. vectors are not sparse"

What if the sparse component has low-rank?

E.g. first column of E is minus that of L

$$E = \begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad M = L + E = \begin{bmatrix} 0 & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & * & \cdots & * & * \end{bmatrix}$$

What if the sparse component has low-rank?

E.g. first column of E is minus that of L

$$E = \begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad M = L + E = \begin{bmatrix} 0 & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & * & \cdots & * & * \end{bmatrix}$$

Sparsity pattern will be assumed (uniform) random

Principal Component Pursuit (PCP)

$$M = L + E$$

- L unknown (rank unknown)
- E unknown (# of entries $\neq 0$, locations, magnitudes all unknown)

Principal Component Pursuit (PCP)

$$M = L + E$$

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Recovery via (convex) PCP

$$\begin{array}{ll} \text{minimize} & \|\hat{L}\|_* + \lambda \|\hat{E}\|_1 \\ \text{subject to} & \hat{L} + \hat{E} = M \end{array}$$

See also Chandrasekaran, Sanghavi, Parrilo, Willsky ('09)

- nuclear norm: $\|L\|_* = \sum_i \sigma_i(L)$ (sum of sing. values)
- ℓ_1 norm: $\|S\|_1 = \sum_{ij} |S_{ij}|$ (sum of abs. values)

Main result: $M = L + E$

Theorem (C., Li, Ma and Wright, 09)

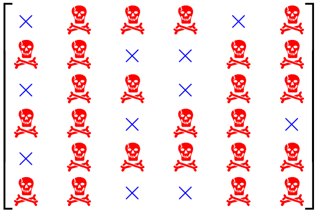
- *L is $n \times n$ of rank*$(L) \leq \rho_r n \, \mu^{-1} (\log n)^{-2}$
- *E is $n \times n$, random sparsity pattern of cardinality $m \leq \rho_s n^2$*

Then with probability $1 - O(n^{-10})$, PCP with $\lambda = 1/\sqrt{n}$ is exact:

$$\hat{L} = L, \quad \hat{E} = E$$

Same conclusion for rectangular matrices with $\lambda = 1/\sqrt{\text{max dim}}$

- Exact
  - whatever the magnitudes of L!
  - whatever the magnitudes of E!
- No tuning parameter!



Connections with matrix completion (MC)

Missing vs. corrupted data

$$\begin{bmatrix} \times & ? & ? & ? & \times & ? \\ ? & ? & \times & \times & ? & ? \\ \times & ? & ? & \times & ? & ? \\ ? & ? & \times & ? & ? & \times \\ \times & ? & ? & ? & ? & ? \\ ? & ? & \times & \times & ? & ? \end{bmatrix}$$

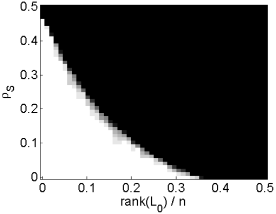
MC: missing

$$\begin{bmatrix} \times & \skull & \skull & \skull & \times & \skull \\ \skull & \skull & \times & \times & \skull & \skull \\ \times & \skull & \skull & \times & \skull & \skull \\ \skull & \skull & \times & \skull & \skull & \times \\ \times & \skull & \skull & \skull & \skull & \skull \\ \skull & \skull & \times & \times & \skull & \skull \end{bmatrix}$$

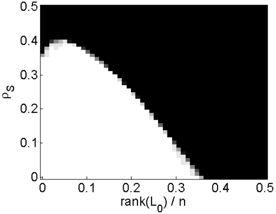
RPCA: corrupted

<u>Harder</u> to detect and correct than to fill in

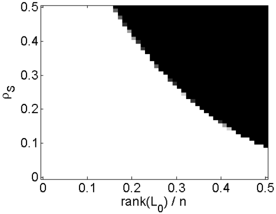
Phase transitions in probability of success



(a) PCP, Random Signs



(b) PCP, Coherent Signs



(c) Matrix Completion

$L = XY^*$ is a product of independent $n \times r$ i.i.d. $\mathcal{N}(0, 1/n)$ matrices

Other works

<u>Chandrasekaran, Sanghavi, Parrilo and Willsky (09)</u>: deterministic results

- Hsu, Kakade and Zhang (10)
- Chen, Jalali, Sanghavi and Caramanis (11)
- Li (11)

Tying it together

PCP

$$
\begin{array}{ll}
\text{min} & \|\hat{L}\|_* + \lambda \|\hat{E}\|_1 \\
\text{s. t.} & \hat{L}_{ij} + \hat{E}_{ij} = L_{ij} + E_{ij} \ \ (i,j) \in \Omega_{\text{obs}}
\end{array}
$$

$$
\begin{bmatrix}
\times & \skull & ? & ? & \times & ? \\
? & ? & \times & \skull & ? & ? \\
\times & ? & ? & \times & ? & ? \\
? & ? & \times & ? & ? & \skull \\
\times & ? & \skull & ? & ? & ? \\
? & ? & \times & \skull & ? & ?
\end{bmatrix}
$$

Tying it together

PCP

$$\begin{array}{ll} \text{min} & \|\hat{L}\|_* + \lambda \|\hat{E}\|_1 \\ \text{s. t.} & \hat{L}_{ij} + \hat{E}_{ij} = L_{ij} + E_{ij} \;\; (i,j) \in \Omega_{\text{obs}} \end{array}$$

Theorem (C., Li, Ma and Wright, 09)

- *L as before, rank$(L) \leq \rho_0 \, n \, \mu^{-1} (\log n)^{-2}$*
- *Ω_{obs} random set of size $m = 0.1 n^2$ (missing frac. is arbitrary)*
- *Each observed entry corrupted with prob. $\tau \leq \tau_0$*

Then with prob. $1 - O(n^{-10})$, PCP with $\lambda = 1/\sqrt{0.1n}$ is exact:

$$\hat{L} = L$$

Same conclusion for rectangular matrices with $\lambda = 1/\sqrt{0.1 \text{max dim}}$

Gross errors + noise

Extension (C., Li, Ma, Wright & Zhou '10)

$$Y_{ij} = L_{ij} + E_{ij} + Z_{ij} \quad (i,j) \in \Omega$$

- L low rank
- E sparse (gross errors)
- Z stochastic or deterministic perturbation

Gross errors + noise

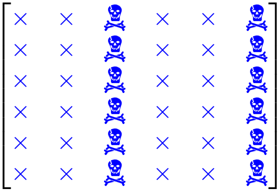
Extension (C., Li, Ma, Wright & Zhou '10)

$$Y_{ij} = L_{ij} + E_{ij} + Z_{ij} \quad (i,j) \in \Omega$$

- L low rank
- E sparse (gross errors)
- Z stochastic or deterministic perturbation

PCP with relaxed constraints $\Rightarrow$ error as if no impulsive noise

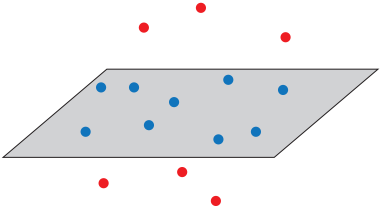
Other models: Xu, Caramanis & Sanghavi '10



Observe all entries of Y

$$Y = L + C \ (+Z)$$

- L low rank
- C column of outliers
- Z stochastic or deterministic perturbation



Goal

Achieve segmentation (noiseless case):

- Identify columns in low-dim subspace
- Identify outliers

Computational issues

Wish to solve the SDP

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|E\|_1 \\ \text{subject to} & L + E = M \end{array}$$

- Off-the-shelf algorithms (SDPT3, SeDuMi) need $n < 80, 100$
- Customized IPMs don't do much better

Have developed a simple and scalable algorithm via the Alternating Direction Method of Multipliers (ADMM)

Empirical performance

n	$\text{rank}(L)$	$\|E\|_0$	$\text{rank}(\hat{L})$	$\|\hat{E}\|_0$	$\frac{\|\hat{L}-L\|_F}{\|L\|_F}$	# SVD	Time(s)
500	25	12,500	25	12,500	1.1×10^{-6}	16	2.9
1,000	50	50,000	50	50,000	1.2×10^{-6}	16	12.4
2,000	100	200,000	100	200,000	1.2×10^{-6}	16	61.8
3,000	250	450,000	250	450,000	2.3×10^{-6}	15	185.2

rank$(L) = 0.05 \times n$, $\|E\|_0 = 0.05 \times n^2$.

n	$\text{rank}(L)$	$\|E\|_0$	$\text{rank}(\hat{L})$	$\|\hat{E}\|_0$	$\frac{\|\hat{L}-L\|_F}{\|L\|_F}$	# SVD	Time(s)
500	25	25,000	25	25,000	1.2×10^{-6}	17	4.0
1,000	50	100,000	50	100,000	2.4×10^{-6}	16	13.7
2,000	100	400,000	100	400,000	2.4×10^{-6}	16	64.5
3,000	150	900,000	150	900,000	2.5×10^{-6}	16	191.0

rank$(L) = 0.05 \times n$, $\|E\|_0 = 0.10 \times n^2$.

Computational cost higher than classical PCA but not by a large factor!

Implementation status

$n \times n$ matrices

- Implementation on desktop for $n \sim 10^3, 10^4$
- Distributed implementation for $n \sim 10^6$ on Redmond HPC clusters (MSRA)
- Support applications with real high-dim. data
  - images
  - videos
  - audio
  - text documents
  - ...

Some Applications

Application to video surveillance

Sequence of 200 video frames (144×172 pixels) with a static background

Problem: detect any activity in the foreground

Background modeling from surveillance video, I

(a) Original (b) Low-rank $\hat{L}$ (c) Sparse $\hat{E}$ (d) Low-rank $\hat{L}$ (e) Sparse $\hat{E}$

PCP Alternating minimization

Alternating minimization of an M-estimator (De La Torre and Black, '03)

Background modeling from surveillance video, II

(a) Original (b) Low-rank $\hat{L}$ (c) Sparse $\hat{E}$ (d) Low-rank $\hat{L}$ (e) Sparse $\hat{E}$

PCP Alternating minimization

Three frames from a 250 frame sequence taken in a lobby, with varying illumination (Li et al., '04).

APPLICATIONS – *Repairing vintage movies*

Original D

Repaired A

Corruptions

Frame 1

480×620 pixels

APPLICATIONS – *Repairing vintage movies*

Original D **Repaired** A

Corruptions

Frame 2

APPLICATIONS – *Repairing vintage movies*

Original D

Corruptions

Repaired A

Frame 3

APPLICATIONS – *Repairing vintage movies*

Original D

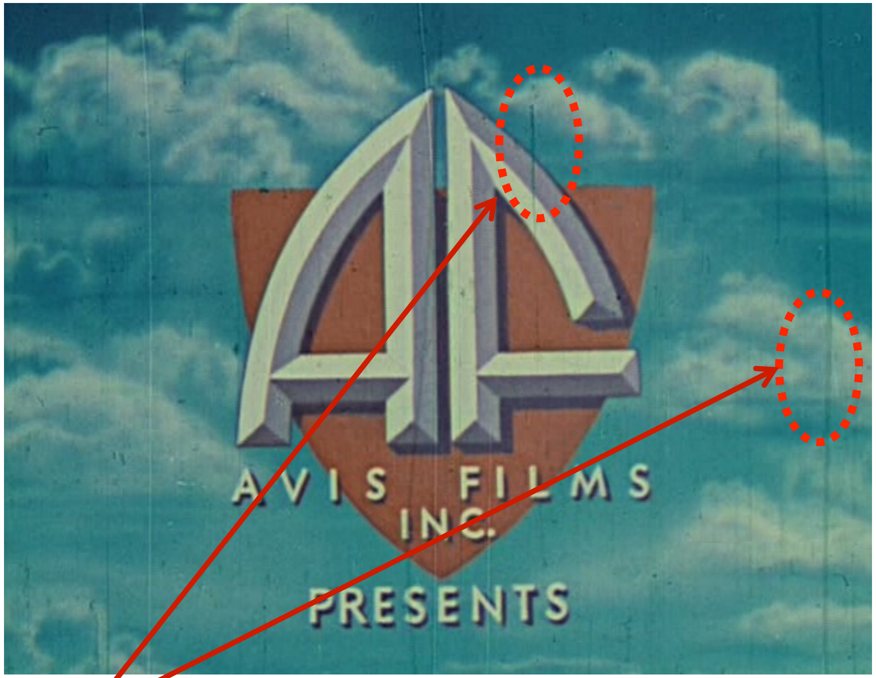
Corruptions

Repaired A

Frame 4

APPLICATIONS – *Repairing vintage movies*

Original D



Repaired A

Corruptions

Frame 5

APPLICATIONS – *Repairing vintage movies*

Original D

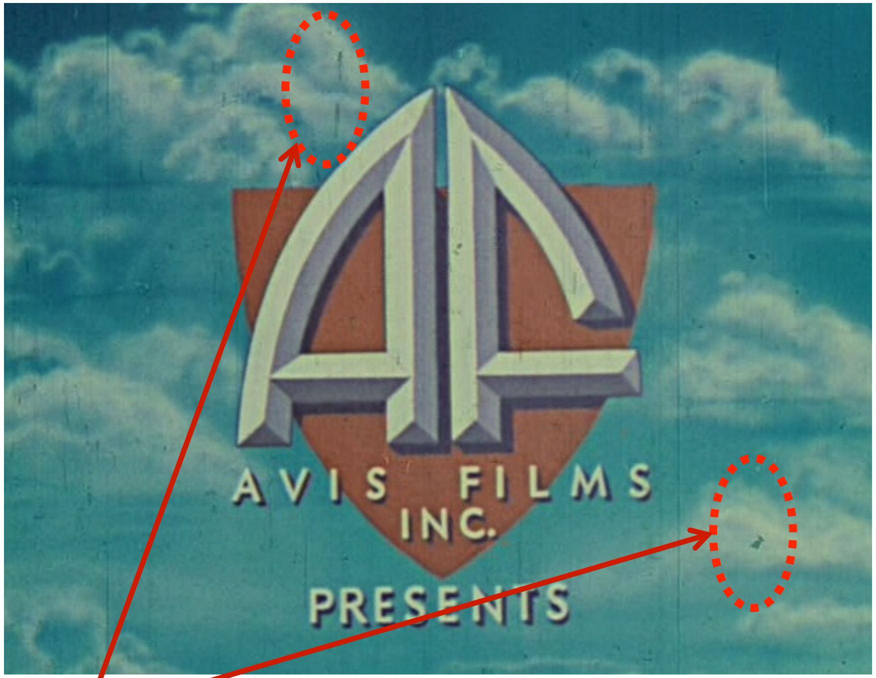
Repaired A

Corruptions

Frame 6

APPLICATIONS – *Repairing vintage movies*

Original D



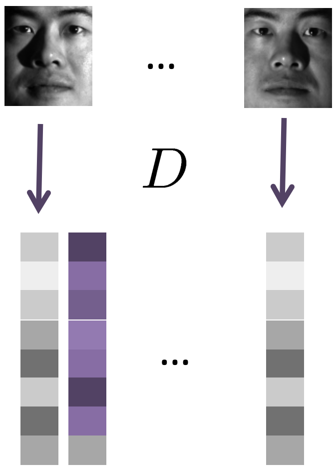
Corruptions

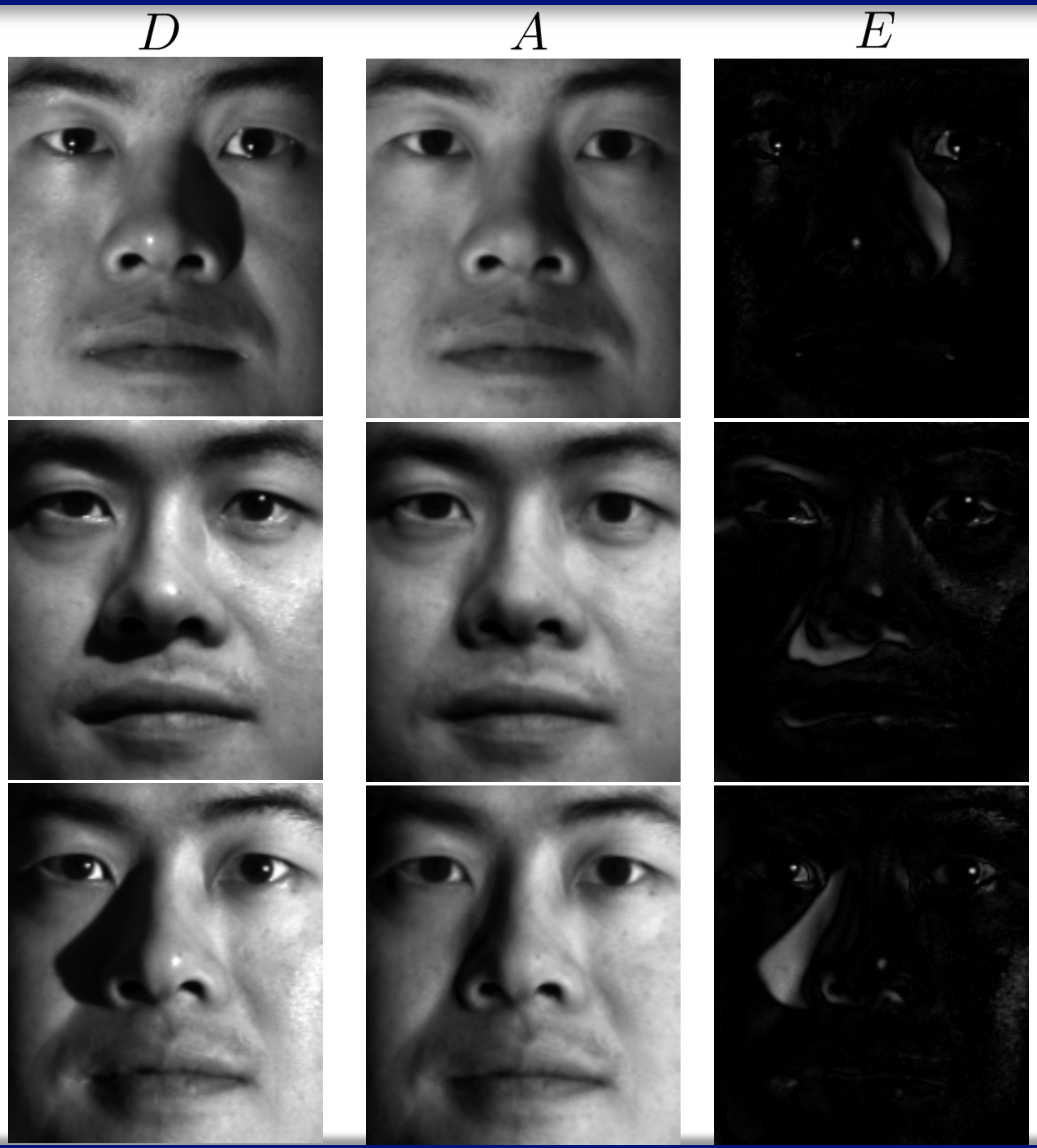
Repaired A

Frame 7

APPLICATIONS – *Faces under varying illumination*

58 images of one person under varying lighting:





APPLICATIONS – *Faces under varying illumination*

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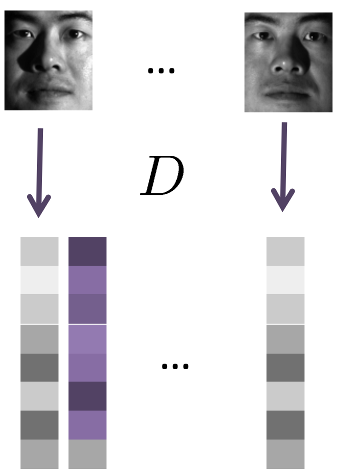

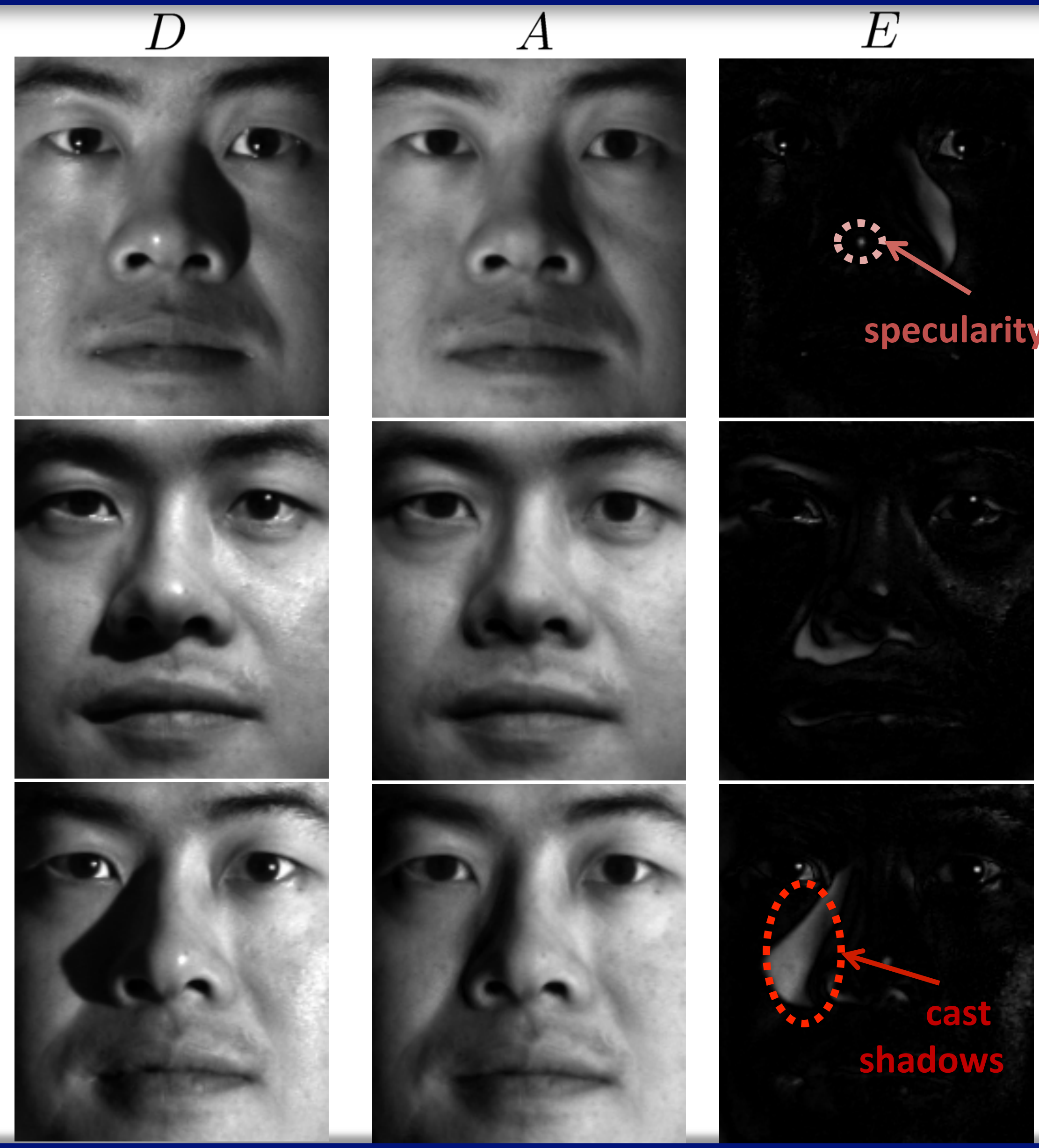

Robust batch image alignment (Ma et al.)

- *Input*: $\boldsymbol{M}$ corrupted and misaligned batch of images (data)
- *Output*: $\boldsymbol{L}$ aligned low-rank images; $\boldsymbol{S}$ sparse errors

$$\text{(Model)} \qquad \boldsymbol{M} \circ \tau = \boldsymbol{L_0} + \boldsymbol{S_0}$$

τ: parametric deformation (rigid, affine, projective)

Robust batch image alignment (Ma et al.)

- *Input*: $\boldsymbol{M}$ corrupted and misaligned batch of images (data)
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$$\text{(Model)} \qquad \boldsymbol{M} \circ \tau = \boldsymbol{L_0} + \boldsymbol{S_0}$$

τ: parametric deformation (rigid, affine, projective)

Bootstrap: find $\boldsymbol{L}$ and $\boldsymbol{S}$ and τ solution to

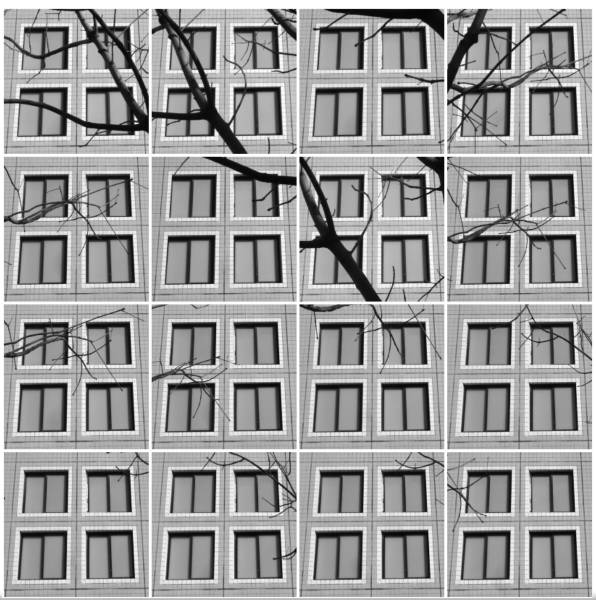
$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}\|_1 \\ \text{subject to} & \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{M} \circ \tau \end{array}$$

APPLICATIONS – *2D image matching and 3D modeling*

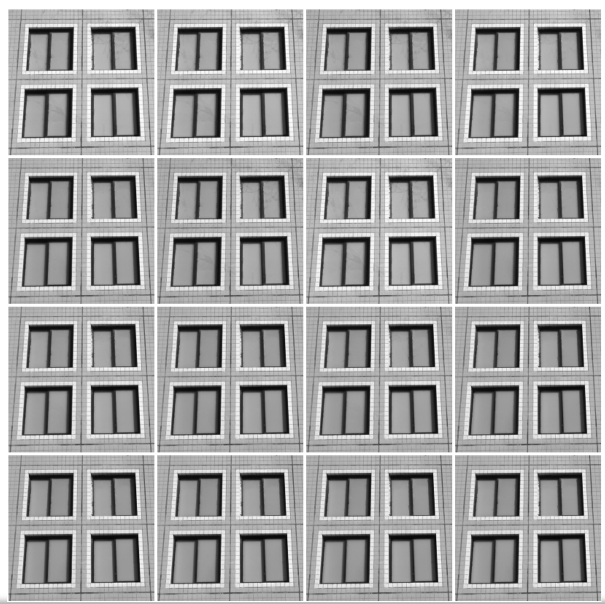
D

$\tau \in$ 2D homographies

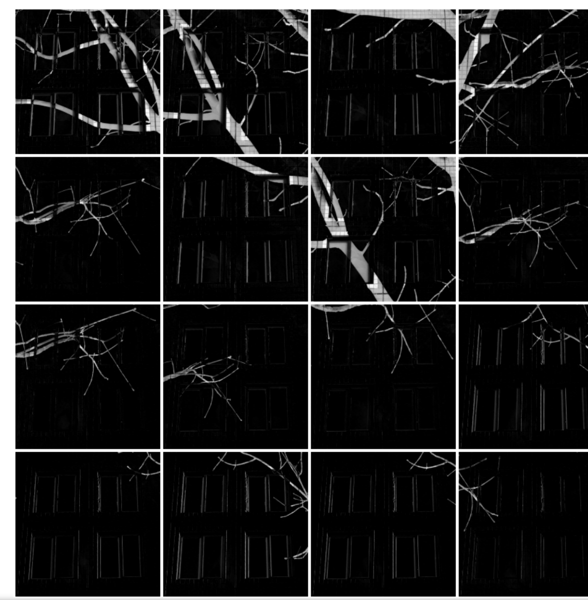
$D \circ \tau$



A



E



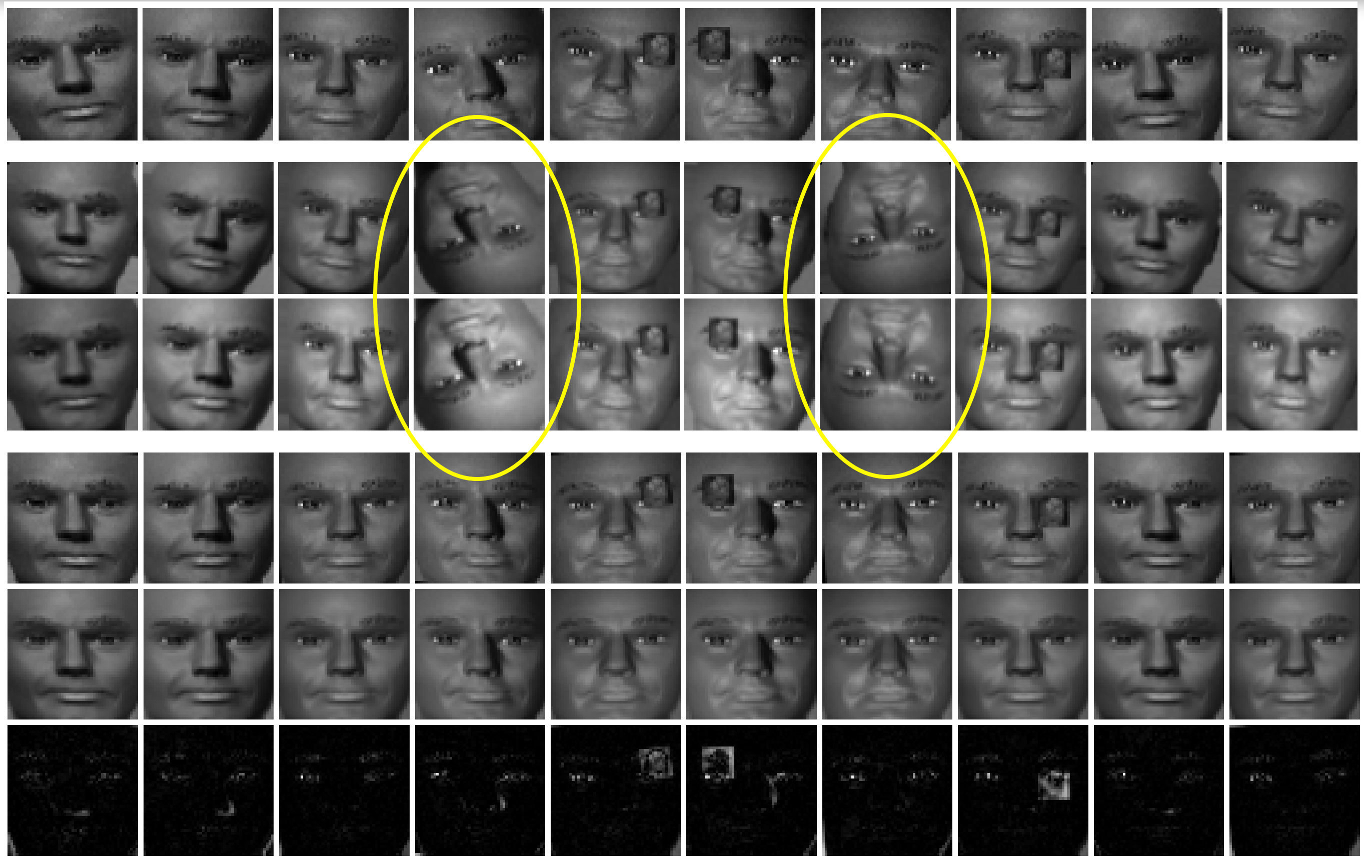
Peng, Ganesh, Wright, Ma, CVPR’10

APPLICATIONS – *Batch face alignment: accuracy evaluation*

100 misaligned corrupted images:

Vedaldi CVPR'08 direct/gradient

RASL:



	Mean error	Error std.	Max error
Initial misalignment	2.5	1.03	4.87
Vedaldi (direct/gradient)	1.97/1.66	1.11/0.85	5.71/4.02
RASL (this work)	0.48	0.23	1.07

APPLICATIONS – *Simultaneous Alignment and Repairing*

$D \circ \tau$

A

E

Peng, Ganesh, Wright, Ma, CVPR'10

APPLICATIONS – *Celebrities from the Internet*

Average face before alignment & repairing

Gloria Macapagal Arroyo
Jennifer Capriati
Laura Bush
Serena Williams
Barack Obama
Ariel Sharon
Arnold Schwarzenegger
Colin Powell
Donald Rumsfeld
George W Bush
Gerhard Schroeder
Hugo Chavez
Jacques Chirac
Jean Chretien
John Ashcroft
Junichiro Koizumi
Lleyton Hewitt
Luiz Inacio Lula da Silva
Tony Blair
Vladimir Putin

APPLICATIONS – *Face recognition with less controlled data?*

Average face after alignment & repairing

Gloria Macapagal Arroyo
Jennifer Capriati
Laura Bush
Serena Williams
Barack Obama
Ariel Sharon
Arnold Schwarzenegger
Colin Powell
Donald Rumsfeld
George W Bush
Gerhard Schroeder
Hugo Chavez
Jacques Chirac
Jean Chretien
John Ashcroft
Junichiro Koizumi
Lleyton Hewitt
Luiz Inacio Lula da Silva
Tony Blair
Vladimir Putin

APPLICATIONS – *Aligning handwritten digits*

D

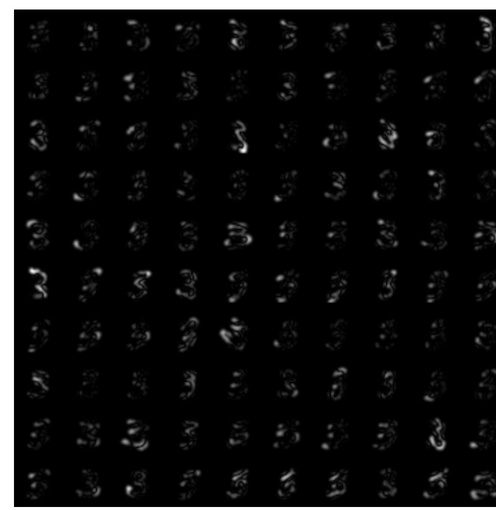
Learned-Miller PAMI'06

Vedaldi CVPR'08

$D \circ \tau$

A

E

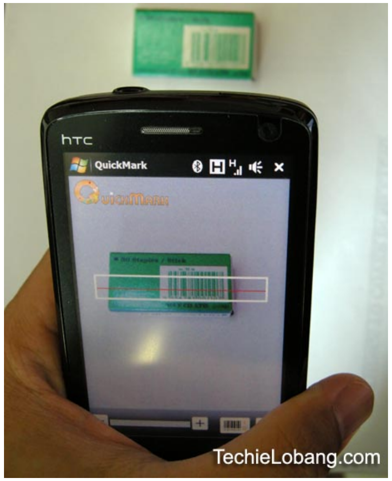


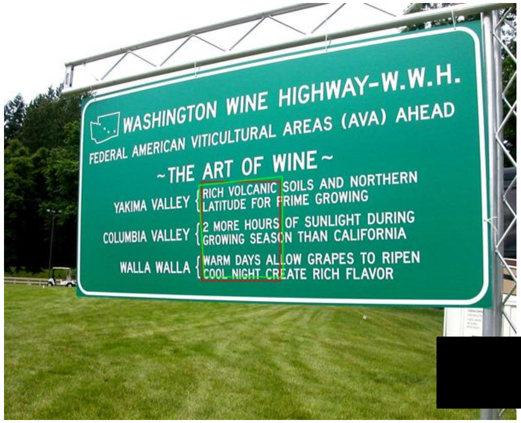
The world we see (through camera) is tilted!

augmented reality

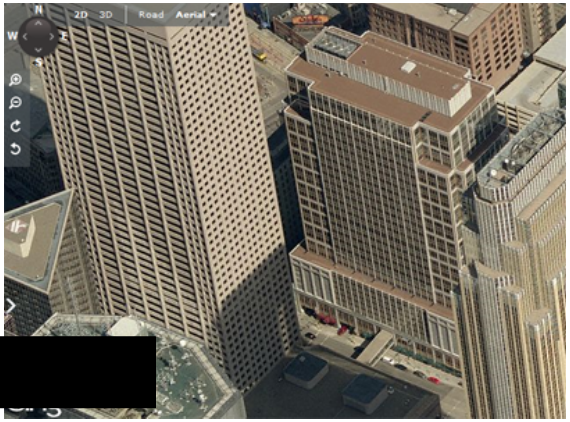
world lens

3D map





[redacted]

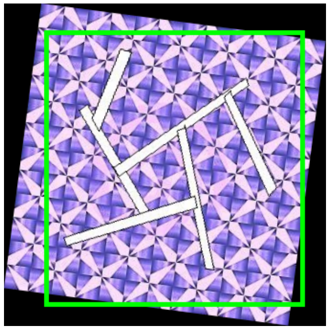
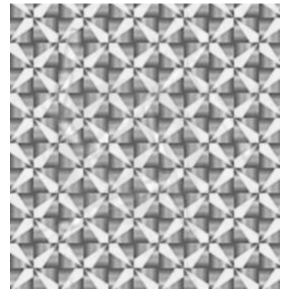
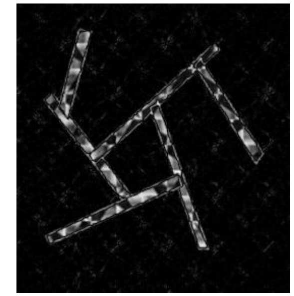


Transform Invariant Low-rank Textures (TILT)

D– corrupted & deformed observation

A– rectified low-rank textures

E– sparse errors

 $\circ\, \tau =$  $+$ 

Problem: Given $D \circ \tau = A_0 + E_0$, recover τ, A_0, E_0

Parametric deformations (affine, projective…)

Low-rank component

Sparse component

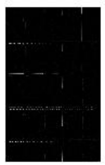
Solution: estimate the deformation and low-rank texture simultaneously

Iteratively solving the linearized convex program:

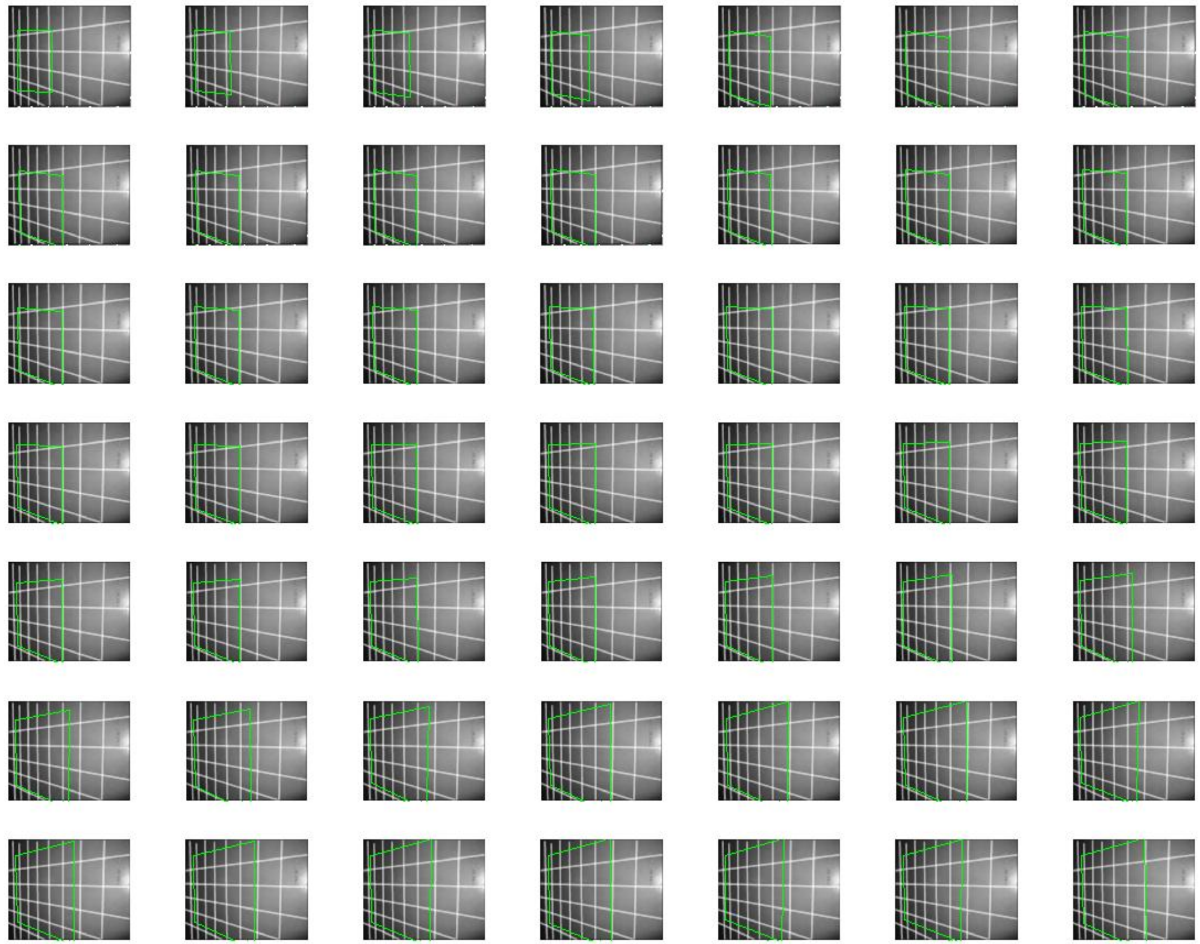
$$\min \ \|A\|_* + \lambda \|E\|_1 \quad \text{subj} \quad A + E = D \circ \tau_k + J \Delta\tau$$

TILT via Iterative RPCA-Like Convex Optimization

D  $D \circ \tau$ 

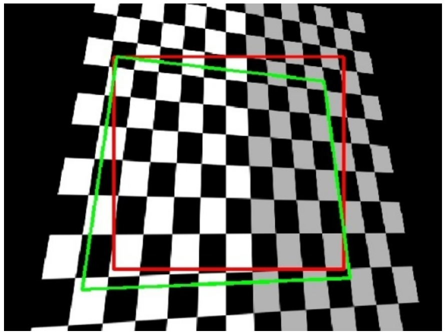
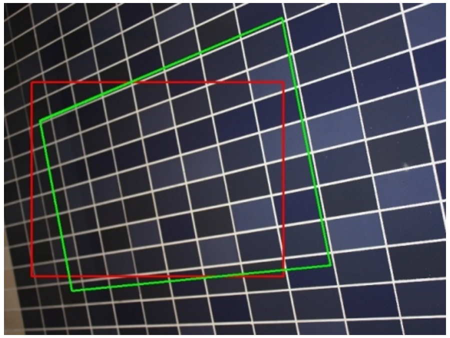
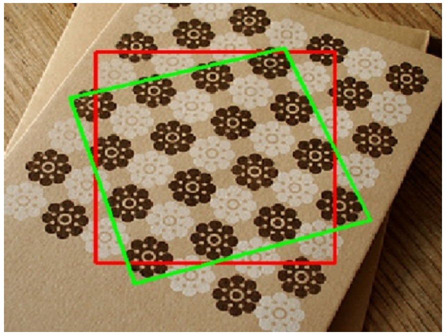
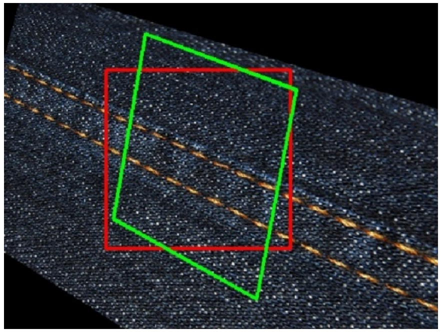
A  E 

Iteration Processes

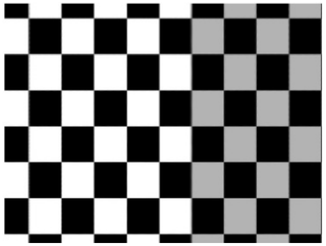
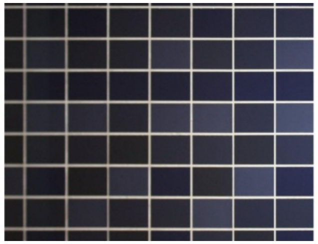


TILT: Examples of Symmetric Patterns and Textures

Input (red window)

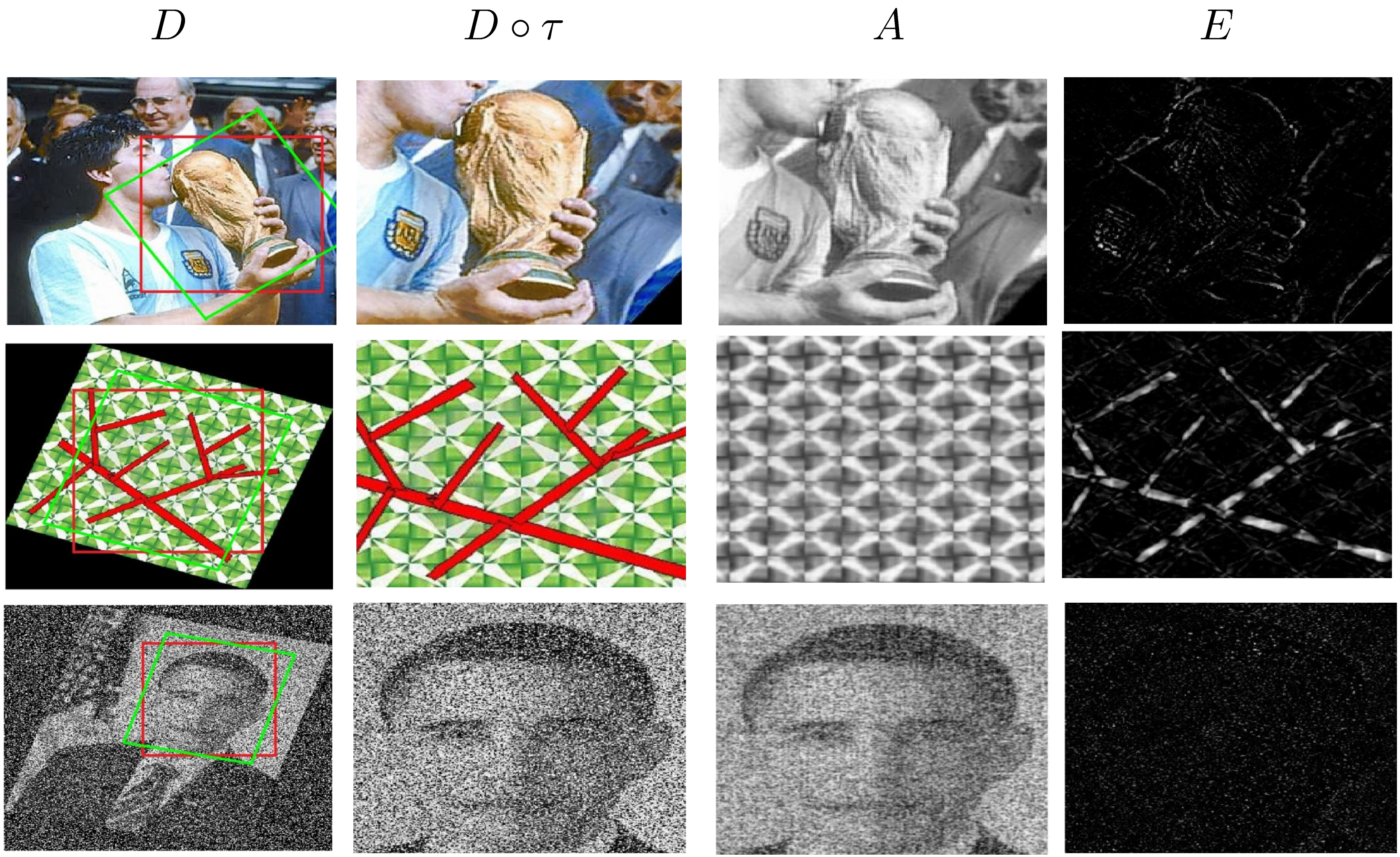





Output (rectified green window)

Zhang, Liang, Ganesh, Ma, ACCV'10

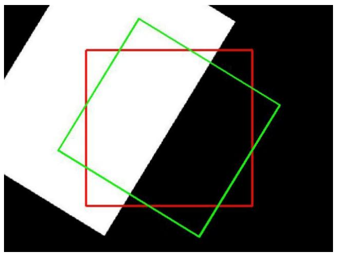
TILT – Robust to Background, Occlusion, and Corruption



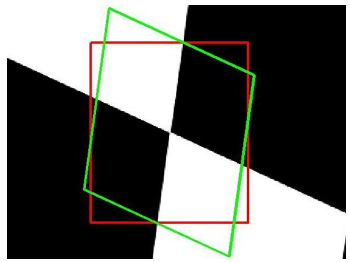
Zhang, Liang, Ganesh, Ma, ACCV'10

TILT: All Types of Regular Geometric Structures in Images

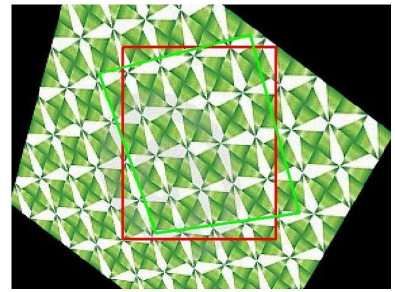
an ideal edge



an ideal corner

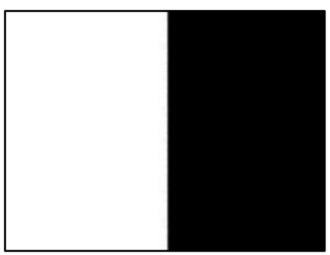


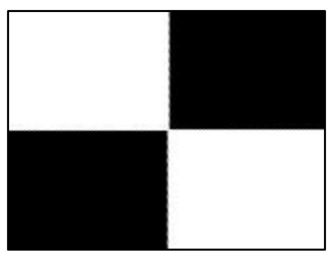
symmetry

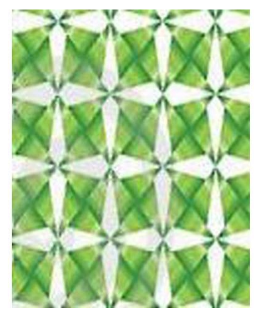


man-made

Rectified Low-rank Textures

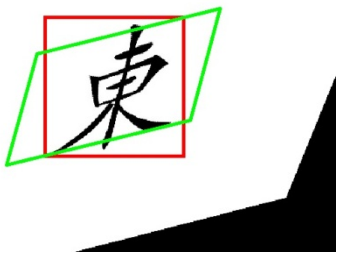


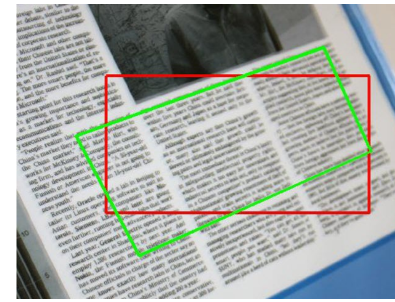




TILT: Examples of Characters, Signs, and Texts

Input (red window)

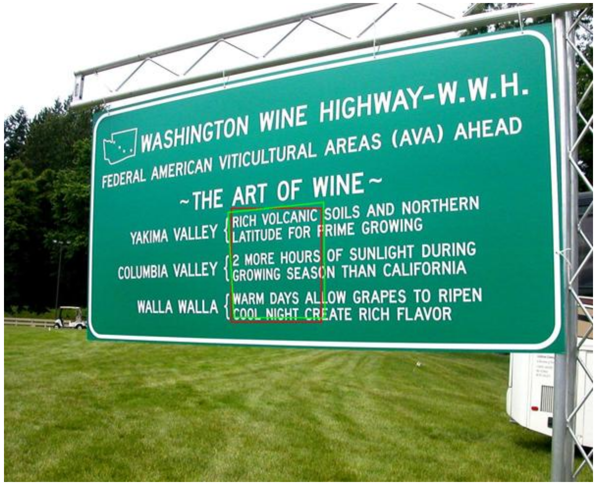


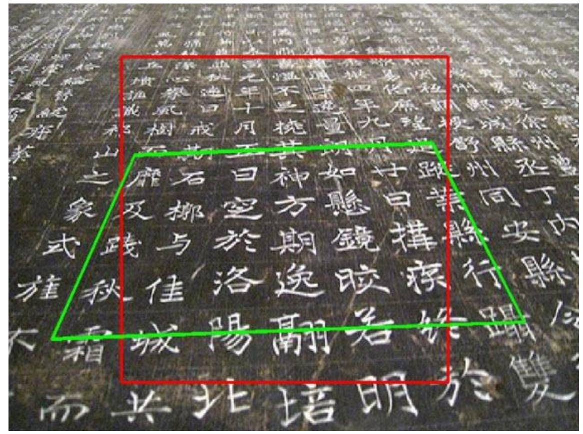


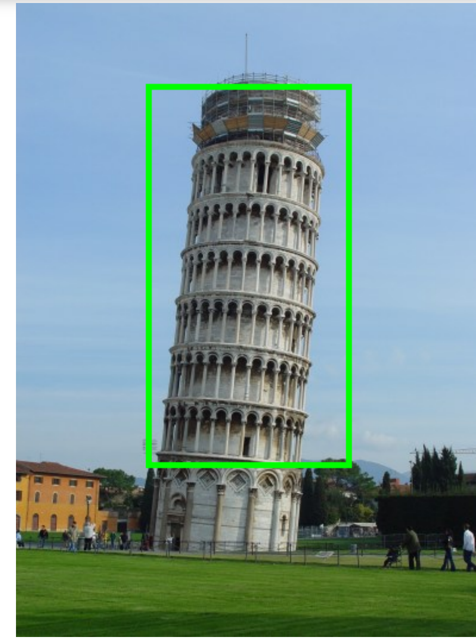
Output (rectified green window)

TILT: More Examples

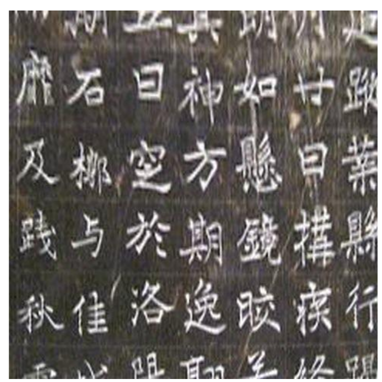
Input (red window)





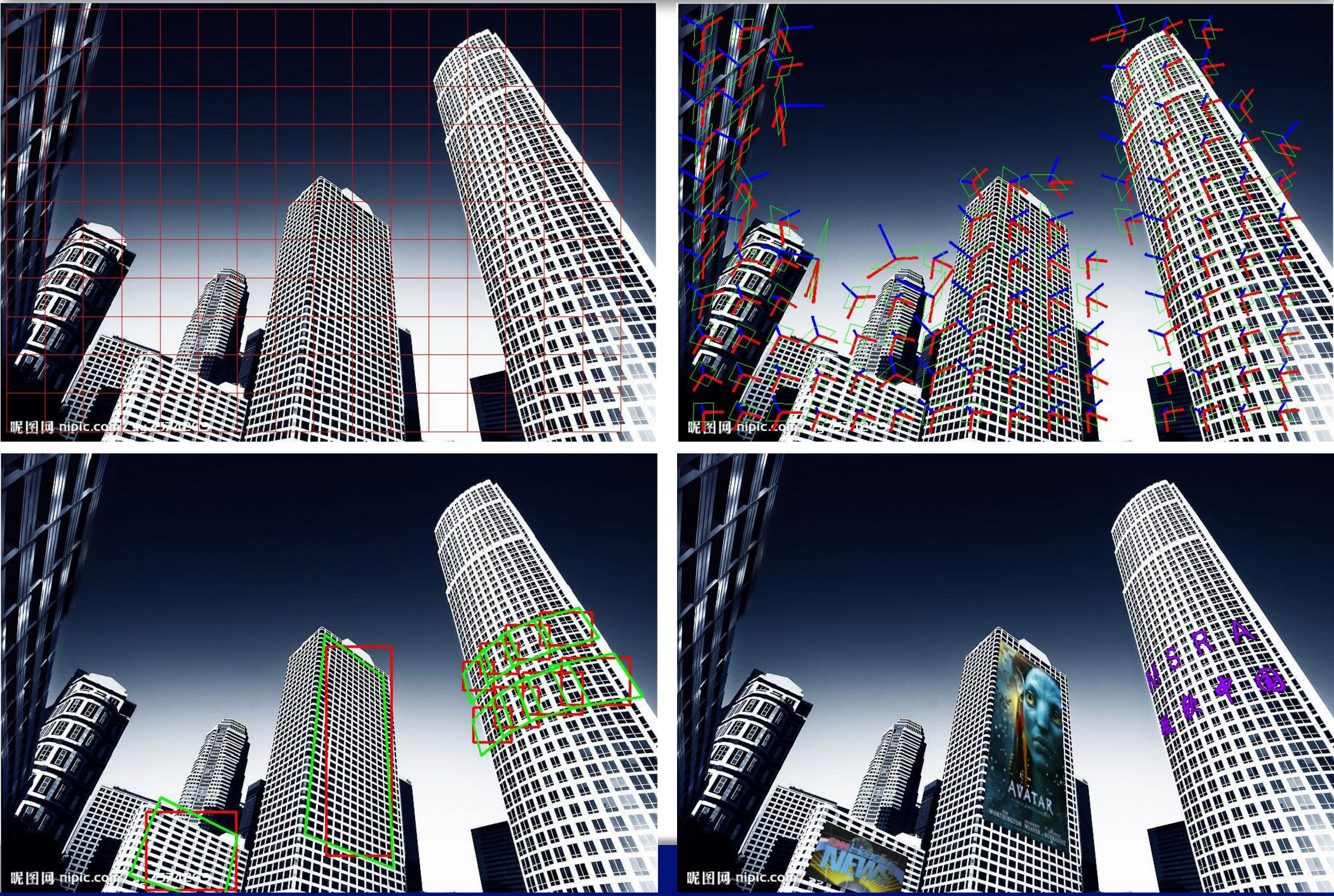


Output (rectified green window)



TILT – 3D Geometry from a Single Image

TILT Applications: Augmented Reality

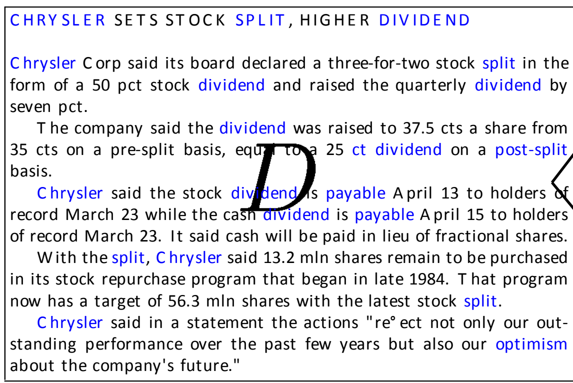


Other Applications: Web Document Corpus Analysis

Latent Semantic Indexing: the classical solution (PCA)

Documents

Words


CHRYSLER SETS STOCK SPLIT, HIGHER DIVIDEND

Chrysler Corp said its board declared a three-for-two stock split in the form of a 50 pct stock dividend and raised the quarterly dividend by seven pct.

The company said the dividend was raised to 37.5 cts a share from 35 cts on a pre-split basis, equal to a 25 ct dividend on a post-split basis.

Chrysler said the stock dividend is payable April 13 to holders of record March 23 while the cash dividend is payable April 15 to holders of record March 23. It said cash will be paid in lieu of fractional shares.

With the split, Chrysler said 13.2 mln shares remain to be purchased in its stock repurchase program that began in late 1984. That program now has a target of 56.3 mln shares with the latest stock split.

Chrysler said in a statement the actions "re° ect not only our outstanding performance over the past few years but also our optimism about the company's future."

D

d_{ij} word frequency (or TF/IDF)

$$D = A + Z$$
$$= U_1 \Sigma_1 V_1^T + \underline{U_2 \Sigma_2 V_2^T}$$

Dense, difficult to interpret

a better model/solution?

$$D = A + \underline{E}$$

Low-rank
“background”
topic model

Low dimensional topic models with keywords…

Other Applications: Sparse Keywords Extracted

Reuters-21578 dataset: 1,000 longest documents; 3,000 most frequent words

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Summary

- Lots of exciting work in theory of low-rank models (matrix completion)
- Lots more needs to be done