CSC2100B Data Structures
Recurrence Relations

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Recurrence Relations

- Recurrence relations are useful in certain counting problems.
- A recurrence relation relates the n-th element of a sequence to its predecessors.
- Recurrence relations arise naturally in the analysis of recursive algorithms.

Sequences and Recurrence Relations

- A (numerical) sequence is an ordered list of number.
  - 2, 4, 6, 8, … (positive even numbers)
  - 0, 1, 1, 2, 3, 5, 8, … (the Fibonacci numbers)
  - 0, 1, 3, 6, 10, 15, … (numbers of key comparisons in selection sort)

Definitions

- **A recurrence relation for the sequence, a_0, a_1, ... is an equation that relates an to certain of its predecessors a_0, a_1, ... , a_{n-1}.**
- **Initial conditions for the sequence a_0, a_1, ... are explicitly given values for a finite number of the terms of the sequence.**

Example

- **A person invests $1000 at 12% compounded annually. If A_n represents the amount at the end of n years, find a recurrence relation and initial conditions that define the sequence A_n.**

- **At the end of $n-1$ years, the amount is A_{n-1}. After one more year, we will have the amount A_{n-1} plus the interest. Thus $A_n = A_{n-1} + (0.12)A_{n-1} = (1.12)A_{n-1}, n \geq 1$.**

- **To apply this recurrence relation for $n = 1$, we need to know the value of A_0 which is 1000.**

Solving Recurrence Relations

- **Iteration - we use the recurrence relation to write the n-th term an in terms of certain of its predecessors $a_{n-1}, \ldots, a_0$.**
- **We then successively use the recurrence relation to replace each of $a_{n-1}, \ldots$ by certain of their predecessors.**
- **We continue until an explicit formula is obtained.**

Some Definitions of Linear Second-order recurrences with constant coefficients

- kth-order
  - Elements x(n) and x(n-k) are k positions apart in the unknown sequence.
- Linear
  - It is a linear combination of the unknown terms of the sequence.
- Constant coefficients
  - The assumption that a, b, and c are some fixed numbers.
- Homogeneous
  - If f(x) = 0 for every n.

Solving Recurrence Relations

- Linear homogeneous recurrence relations with constant coefficients - a linear homogeneous recurrence relation of order k with constant coefficients is a recurrence relation of the form

$$a_0 = c_0, a_1 = c_1, \ldots, a_{k-1} = c_{k-1}$$

- Notice that a linear homogeneous recurrence relation of order K with constant coefficients, together with the k initial conditions

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}, c_k \geq 0$$

uniquely defines a sequence $a_0, a_1, \ldots$

Example

- Nonlinear

$$a_n = 3a_{n-1}a_{n-2}.$$

- Inhomogeneous

$$a_n - a_{n-1} = 2n.$$

- Homogeneous recurrence relation with nonconstant coefficients

$$a_n = 3n \cdot a_{n-1}.$$

Iteration Example

- **We can solve the recurrence relation** $a_n = a_{n-1} + 3$ **subject to the initial condition** $a_1 = 2$**, by iteration.**
- $a_{n-1} = a_{n-2} + 3.$
- $a_n = a_{n-1} + 3 = a_{n-2} + 3 + 3 = a_{n-2} + 2 \times 3.$
- $a_{n-2} = a_{n-3} + 3.$
- $a_n = a_{n-2} + 2 \times 3 = a_{n-3} + 3 + 2 \times 3 = an - 3 + 3 \times 3.$
- $a_n = a_{n-k} + k \times 3 = 2 + 3(n-1).$

Iteration Example

- **In general, to solve** $a_n = a_{n-1} + k, a_1 = c$, **one obtains** $a_n = c + k(n-1)$.
- **We can solve the recurrence relation**
  - $a_n = ka_{n-1}, a_0 = c$.
  - $a_n = ka_{n-1} = k(ka_{n-2}) = \ldots = k^n a_0 = ck^n$.

Linear Homogeneous Recurrence Example

$$a_n = 5a_{n-1} - 6a_{n-2}, a_0 = 7, a_1 = 16$$

- **Since the solution was of the form $a_n = t^n$, thus for our first attempt at finding a solution of the second-order recurrence relation, we will search for a solution of the form $a_n = t^n$.**
  - $t^n = 5t^{n-1} - 6tn - 2$
  - $t^2 - 5t + 6 = 0.$

Example

- **Solving the above we obtain, $t = 2, t = 3$.**
- **At this point, we have two solutions S and T given by**
  - $S_n = 2^n, T_n = 3^n$.
- **We can verify that is S and T are solutions of the above, then $bS + dT$, where b and d are any numbers whatever, is also a solution of the above.**

Example

- **In our case, if we define the sequence U by the equation**
  - $U_n = bS_n + dT_n$
  - $= b2^n + d3^n$
- **To satisfy the initial conditions, we must have**
  - $7 = U_0 = b2^0 + d3^0 = b + d$.
  - $16 = U_1 = b2^1 + d3^1 = 2b + 3d$.

Example

- **Solving these equations for b and d, we obtain**
  - $b = 5, d = 2$
- **Therefore, the sequence U defined by**
  - $U_n = 5 \times 2^n + 2 \times 3^n$

satisfies the recurrence relation and the initial conditions.

Fibonacci Sequence

- **The Fibonacci sequence is defined by the recurrence relation**
  - $f_n = f_{n-1} + f_{n-2}, n \geq 3$ **and initial conditions**
  - $f_1 = 1, f_2 = 2$**.**
- **We begin by using the quadratic formula to solve**
  - $t^2 - t - 1 = 0$
- **The solutions are**
  - $t = \frac{1 \pm \sqrt{5}}{2}$

Example

- Thus the solution is of the form

$$f_n = b(\tfrac{1+\sqrt{5}}{2})^n + d(\tfrac{1-\sqrt{5}}{2})^n.$$

- To satisfy the initial conditions, we must have

$$b(\tfrac{1+\sqrt{5}}{2}) + d(\tfrac{1-\sqrt{5}}{2}) = 1,$$

$$b(\tfrac{1+\sqrt{5}}{2})^2 + d(\tfrac{1-\sqrt{5}}{2})^2 = 2.$$

Example

- Solving these equations for b and d, we obtain

$$b = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2}), d = -\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2}).$$

- Therefore, an explicit formula for the Fibonacci sequence is

$$f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1}.$$

Tower of Hanoi

- **Find an explicit formula for a_n, the minimum number of moves in which the n-disk Tower of Hanoi puzzle can be solved.**

- $a_n = 2a_{n-1} + 1, a_1 = 1.$

- **Applying the iterative method, we obtain**

$$
\begin{aligned}
a_n &= 2a_{n-1} + 1 \\
&= 2(2a_{n-2} + 1) + 1 \\
&= 2^2 a_{n-2} + 2 + 1 \\
&= 2^2(2a_{n-3} + 1) + 2 + 1 \\
&= 2^3 a_{n-3} + 2^2 + 2 + 1 \\
&\ldots \\
&= 2^{n-1} a_1 + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^n - 1
\end{aligned}
$$

Common Recurrence Types

- **Decrease-by-one**
  - $T(n) = T(n-1) + f(n)$
- **Decrease-by-a-constant-factor**
  - $T(n) = T(n/b) + f(n)$
- **Divide-and-conquer**
  - $T(n) = aT(n/b) + f(n)$