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*Metaphorical Reasoning: Origins, Uses, Development and Interactions in Mathematics**

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People's innate neurological perceptions, mental simulations, intuitions, and schemas provide the familiar source entities, relationships, and actions for a metaphor. People metaphorically project this information on to the target problem to construct new concepts, relationships and actions. People also reason metaphorically to: connect mathematical ideas, improve recall, understand mathematical representations, and enhance their computational environment. Metaphorical reasoning capacity depends not on age but on people's understanding of the source which provides the potential for metaphorical reasoning. Use of a particular metaphor decreases increasing target understanding. A metaphor's source-to-target projections specify the actual metaphorical inferences. Metaphorical reasoning may serve as a permanent resource rather than a temporary scaffold as experts automate their metaphorical reasoning for routine problems. Overlapping metaphors increase coherence of mathematical understanding and compensate for each metaphor's limitations. Facing difficult mathematical relationships in a problem situation, people can create a chain of metaphors from an intuitive source to the mathematics to the problem situation. Finally, metaphorical reasoning differs from the following types of reasoning: embodiments, intersection, analogical, example-based, symbolic play, symbolic mnemonics, and distributed.

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Constructivists argued that students build their own knowledge rather than passively receiving it from others (Piaget, 1952; von Glaserfeld, 1990). To learn new topics, students must use their prior resources. Researchers in the fields of neurology (Barlow & Levick, 1965), psychology (Davidson, 1987; Ogborn & Bliss, 1990), education (diSessa, 1993; Fischbein, 1987) and linguistics (Talmy, 1988) argued that students have access to informal, non-mathematical resources such as knowledge of motion. Using these non-mathematical resources, students can construct mathematical knowledge. Students can project their understanding of the entities, properties, and actions from a familiar *source* phenomena (e.g., boxes or motion) on to a less familiar *target* (e.g., variables) to make sense of it, i.e. *reason metaphorically* (Black, 1993; Lakoff & Johnson, 1980; Lakoff 1987; Nolder, 1991; Pimm, 1987; Presmeg, 1992). For example, a person can understand a variable as a container enclosing a number or as a traveler moving through different numerical locations (see Table 1). By understanding students' metaphorical reasoning, educators can optimally use metaphors to improve students' mathematical competencies.

This article provides a discussion of how students construct metaphorical understanding of mathematics by using their informal resources

Table 1. People Can Metaphorically Understand Variables as Containers or as Travelers

Variables (target)	are	Containers (source 1)	or	Travelers (source 2)
Numbers	←	Objects	or	Locations
Variables represent numbers	←	Boxes contain objects	or	Travelers are at locations
Unknown variable	←	Closed box, object(s) is not visible	or	Unknown location of the traveler
Known variable	←	Open box, object(s) is visible	or	Known location of the traveler
Static variable has only one value	←	Object(s) in a box does not change		—
A variable can represent different numbers at different times		—		Travelers are at different locations at different times
Multiple instances of the same variable always represent the same number	←	The same types of box always contain the same object(s)	or	The same types of travelers are always at the same location at the same time
Different variables can represent different numbers	←	Different types of boxes can contain different object(s)	or	Different types of travelers can be at different locations

with teacher guidance. Then, different benefits of metaphorical reasoning are discussed, followed by a comparison of experts' and novices' metaphorical reasoning with respect to: (a) each metaphor component and (b) multiple metaphors. After contrasting metaphorical reasoning with related types of reasoning, this article concludes with some implications and future research questions.

Constructing Metaphors

Students cannot directly perceive some concepts such as negative numbers in the physical world. So, a teacher must choose a *source* of informal knowledge from which students can metaphorically construct their knowledge of negative numbers. Then, the teacher helps the students create the appropriate metaphorical inferences.

Building Blocks for Metaphorical Reasoning

Using an ecological approach to psychology (Dent, 1990; Gibson, 1979; Turvey, Shaw, Reed, & Mace, 1981), some of the resources that students may use to create a metaphor: neurologically-interpreted perceptions, mental simulations, intuitions, and schemas, are discussed.

Neurologically-Interpreted Perceptions

With their innate brain structures, students interpret basic perceptual stimuli in similar ways. For example, neurological research suggests that a newly-born infant has built-in edge-detectors (Hubel & Wiesel, 1962), color-detectors (Marks, Dobelle, & MacNichol, 1964; Wald, 1959), and motion-detectors (Barlow & Levick, 1965). (See Appendix A for a summary of innate detectors.) People process stimuli from the physical world such as light, edges, and downward motion of falling objects. Because most people interpret external stimuli with similar neurological equipment, their perceptions can serve as overlapping foundations for constructing new knowledge. (Consider how our fundamental understandings of the world would differ if we only perceived magnetic fields.)

Mental Simulations

Human neurology not only detects perceptual stimuli, but also stores them in memory and recalls them. By invoking and recombining prior perceptions, Shepard and Cooper (1982) showed that humans can imagine new situations. Imagine a circle. Color it red. Make it smaller. Move it down. Put a blue triangle inside. These simple *mental simulations* combine different stored perceptions of our physical environment (see Appendix A for a list of mental simulations). For example, Gibson (1950) showed that objects moving away from us appear smaller, activating fewer edge-detectors. Using mental simulations, people can combine past percepts to create mental situations that they have never experienced (albeit with many missing details) such as a 500-foot panda stepping over a bridge.

Intuitions

If particular perceptions or sequences of perceptions recur often, Chiu (1996b) showed that people organize them into self-evident, robust, holistic and conceptual notions, namely *intuitions* (Fischbein's [1987] primary intuitions, but not secondary intuitions). (See Appendix A for a list of intuitions and Regier [1996] for connectionist models of them.) Consider a person who sees a configuration of color(s) and contiguous edges (an object) at one location. Then, that person sees the same configuration at another location a moment later. He or she can infer from frequent past experiences that the object moved (or was moved) from the first location to the second (Chiu's [1996b] *path* intuition). Fischbein (1987) showed that intuitions are self-evident to the individual across different situations. Furthermore, Johnson (1987) argued that a person immediately understands the source, path, and destination aspects together as a single whole. Mandler (1992a, 1992b) showed that applying the intuition requires conceptual thinking beyond perception.

Mandler (1992a, 1992b) argued that infants use *perceptual analysis* to create intuitions. For example, an infant (let us call her Inez) may construct a simple path intuition from early analyses of moving objects. Barlow & Levick's (1965) research suggests that infants have innate motion detectors that indicate that an object has moved (say a falling red ball). Mandler (1992a, 1992b) argued that the infant then consciously organizes a subset of the perceptual stimuli into non-perceptual form, for instance, something red was in one location and is now in another. However the infant's concept is coarse-grained and contains less information than her percepts. Marks, Dobelle, and MacNichol's (1964) and Wald's (1959) research suggests that an infant can capture the red color through her innate cones, but not the ball's shape, texture or the precise path. Later, Inez further develops this path intuition through comparisons with other instances of moving objects. Mandler (1992a, 1992b) distinguishes these intuitions (simple concepts) from percepts through (a) the need for consciously noticing to form concepts, (b) the later accessibility of concepts, and (c) the optional invocation of concepts during perceptual processing.

Schemas

After experiencing and interpreting longer sequences of perceptions and intuitions, people can organize this information into cognitive structures called schemas. Bobrow and Winograd (1977), Minsky (1988), and Schank and Abelson (1977) have shown that people use *schemas* to represent the structure of an object or event according to a slot structure where slots specify values that the object or event has on various attributes. People may recognize that particular combinations of perceptions and intuitions recur frequently, organize these into conceptual relationships, and combine them through mental simulations to create early schemas. (People's schemas and intuitions can likewise influence their perceptual interpretations of stimuli.) These early schemas may consist of composite sequences of perceptions and intuitions with a limited number of slots for different possibilities. (See Appendix A for a selected list of schemas relevant to mathematics.) For example, the infant Inez takes her scarf off and lets go of it. Through her color-detectors, edge-detectors, motion-detectors and her path intuition, she sees it fall to the ground. She picks it up and lets go again. Again it falls to the ground. By repeatedly picking up objects and letting go of them, she learns that objects fall. Furthermore, by doing this with different objects, Inez notices that: some objects fall to the ground and stay there (such as scarves); some fall to the ground and move (such as balls); and others fall and become many smaller pieces (glasses). Composing these patterns together through a mental simulation, she can create a *falling* schema in which a released mental object falls. After it touches the mental ground, there are different possible outcomes.

People reason metaphorically by projecting (via mental simulation) a schema onto either the current situation or on to another schema. Gentner (1989) argues that people initially pair preliminary entities from the source and target. The individual may create these pairings either through self-inspiration or with the guidance of others (e.g., teachers). After these initial pairings, entities and properties related to these initial paired entities are paired with one another. The number and extant of the pairings varies (e.g., depending on the individual's knowledge of the source and target, his or her goals, the situation, etc.).

In short, people can compose neurologically-interpreted perceptions to create mental simulations, combine recurring sequences of perceptions to create intuitions, and integrate intuitions to create schemas. Through mental simulation, a person can then project a schema on to a situation or another schema to reason metaphorically. A person can then combine these elementary perceptions, simulations, intuitions, schemas, and metaphors to create more complex knowledge.

Learning a Metaphor in a Classroom

Nesher (1989) showed how teachers can help students reason metaphorically to learn mathematics through a *learning system* (LS): (a) presenting a system of familiar objects, (b) discussing it with everyday language, and (c) introducing mathematical language. A teacher presents a system of familiar objects, properties and actions (*exemplification component* or source) that both makes sense to the students and can be mapped to the target mathematics. For example, a teacher (let's call her Tanya) orchestrates an activity (walking along a street represented as a line on a blackboard) centered around familiar source entities (starting point, locations, direction, distances, etc.). The activity provides an assessment opportunity for the teacher to ensure that every student has sufficient source understanding to use the metaphor. Students understand the exemplification component through the exemplification objects' immediately perceivable properties (Gibson's [1950] *affordances*) or the students' prior experiences with similar objects. Tanya embeds the *target* mathematics (zero, numbers, sign, absolute value, etc.) into the activity. She juxtaposes them with the source entities, perhaps through a story (school has address zero, addresses of buildings on this street east of school are positive, those west of school are negative, identical distances between adjacent buildings, etc.). This juxtaposition of source and target entities helps a student (let's call him Sean) view target mathematical entities as if they were source entities so that he can project source properties and actions on to the target mathematics.

Students engage in actions with exemplification objects and discuss these actions with their own everyday language to create an *exemplification language*. Then, Tanya asks some questions with everyday language

("If Jose lives at number six east, how do I get to his house from school?"). Using his source understanding, Sean can answer these questions successfully ("walk six blocks to the right"). Then, the teacher gradually introduces mathematical terminology and ways of talking (i.e., a *mathematical language*). Adapting her questions to her students' progress, Tanya presents problems with more target mathematical terminology and fewer references to their source understanding ("starting at negative one, how do I get to two?"). Sean interprets these problems through his familiar source and continues to reason using source relationships and inferences.

Later, when Tanya asks questions strictly with target terminology ("what's negative three plus negative four? [–3 + –4]"), Sean can solve the problem metaphorically by source inferences ("I start at zero [finger points to zero] and walk backwards three blocks [finger bounces three times on the number line from 0 to –3] and then I walk backwards four blocks [finger bounces four times on the number line from –3 to –7] and end up at negative seven"). Students learn to translate their exemplification language descriptions to mathematical language descriptions.

In short, Nesher (1989) argued that students can learn mathematics through making sense of the exemplification component, describing their actions with an exemplification language and finally learning the corresponding mathematical language.

Benefits of Metaphorical Reasoning in Mathematics

Using their intuitive knowledge, students can reason metaphorically to construct new mathematical concepts (as discussed above). In addition, students may use metaphorical reasoning to connect mathematical ideas, improve their memory recall of related mathematical ideas, make sense of mathematical representations, and perform computations.

Integrating Target Concepts

Chiu (1998) showed that students used a metaphor to connect different target concepts together through their relationships in the common source. In that study, graduate students showed their understanding of addition, multiplication and division through metaphors such as ARITHMETIC IS MOTION (see Appendix B for a detailed description of selected metaphors in this article; all metaphors are in small capitals in the form of TARGET IS (←)

SOURCE). According to Gary, a graduate student, "that's [–2 × 3] like turning around and taking two steps of size three, so you get minus three and three [– (3 + 3)], so minus six. You know what I mean? You do the 3 two times but you turn around first." (All names are pseudonyms.) Hence, multiplication is repeated addition (repeated steps). Gary later explained "–8 / –4" by saying "that's two 'cause you do two steps of minus four to get to minus eight, it's the opposite of where you're going, you're seeing how to get there." Therefore, division is the opposite of multiplication. Note how this metaphor can also help students understand the distinction between 4 / 0 and 0 / 0. For 4 / 0, the student starts at location "4" and must return to the origin (0) with steps of size 0, which is impossible. For 0 / 0, the student starts at "0" and can return to "0" with any number of steps of size 0. Because there are an infinite number of answers, not a unique one, the expression "0 / 0" is indeterminate. Thus, metaphorical reasoning can integrate arithmetic operations (+, ×, /) and mathematical notions of impossibility and indeterminateness through source actions and relationships (in this case, motion).

Improving Recall

Reynolds and Schwartz (1983) showed that people encoding information with metaphors recalled more information. Starting with meaningful source connections and projecting them on to the target entities creates connections both between the source and the target and within the target. Consider the formula for the perimeter of a polygon as interpreted through LINES ARE PATHS (perimeter = sum of lengths of all sides of a polygon). To compute the perimeter metaphorically, start at one location and walk along the polygon until you reach your starting point, counting each step along the way. Source-to-target connections include: perimeter is total steps, sides are parts of a path, adjacent sides are adjacent path parts, and length of sides are number of steps to traverse each path part. New target connections include: ordering all sides so that each side has a predecessor and a successor, summing lengths of adjacent sides, and maintaining a partial sum. Anderson (1985) and Reder (1979) showed that when recalling any of these pieces of information, the new connections provide additional means to activate the remaining pieces of information, thereby increasing the likelihood of recalling the algorithm and all the related concepts. By creating additional meaningful connections, metaphorical reasoning can improve recall of mathematical relationships.

Understanding Representations

Metaphorical reasoning also underlies interpretations of many representations as well as mathematical concepts. For the purpose of this article, a representation refers to any perceptually accessible stimuli such as drawings, gestures or talk that an audience interprets. Many mathematical representations such as Venn diagrams, number lines, and graphs rely on metaphorical inferences from a person's source understanding of space. Consider a Venn diagram with one oval (A) completely inside another oval (B), representing set A as a subset of set B. Lakoff (1987) showed that understanding the mathematical relationships in this representation utilizes reasoning through the metaphorical constraints of SETS ARE CONTAINERS and serves as a basis for logical implication and the Boolean logic of classes. Elements of set A are understood as objects inside the container labeled "A." Likewise, elements of set B are understood as objects inside the container labeled "B." Because container A is inside container B, an element inside set A must be inside set B, (A → B). Likewise, an element outside container B must be outside container A, (~B → ~A).

Facilitating Mathematical Computations

Chiu (1996a) showed that metaphorical reasoning can facilitate computations both through metaphorical computations and metaphorical constraints. Middle school students used ARITHMETIC IS MANIPULATING OBJECTS to compute "– 5 + 8." For example, Adam said " Like the negative five are like holes you know [draws five circles], and the, um, eight, positive eight are like marbles [draws eight dots, five inside the circles]. So the holes, um, *eat up* five of them, and so there's three *left*, so the answer's three, positive three." Understanding negative numbers as holes and positive numbers as objects, such as marbles, Adam viewed the problem as combining holes and marbles. By matching each marble to a hole, he computed the result by counting the remaining marbles. Meanwhile, Eva used a metaphorical constraint to solve the same problem, "minus five plus eight is minus three, no wait, the pluses *wipes out* the minuses, so there are only pluses *left* over, so it should be plus, positive three." Viewing the problem as opposing objects, she used a metaphorical constraint (more pluses [8] than minuses [–5]) to detect and correct her initial error. In a computational environment enhanced by metaphorical reasoning, students can generate results through metaphorical computations and detect and correct errors through metaphorical constraints.

In short, students can benefit by using metaphorical reasoning to: understand new concepts, connect mathematical ideas, improve recall, understand mathematical representations, and facilitate computations.

Novice vs. Expert: Source, Target, and Source-to-Target Projections

This section compares how novices and experts in a particular area of mathematics (such as arithmetic) differ with respect to each component of metaphorical reasoning: source knowledge, target knowledge and source-to-target projections.

Differences in Source Comprehension

Metaphorical reasoning capabilities (and learning facility for metaphors) differ primarily along the dimension of experience, not age. Goswami's (1991) review of empirical research showed that metaphorical reasoning develops early and that knowledge differences were better predictors than age differences in metaphorical reasoning. In particular, Chen, Sanchez, and Campbell (1997) showed that infants as young as 13-month olds can reason metaphorically. A person's capacity for reasoning through a particular metaphor depends primarily on the person's understanding of the metaphor's source component, not on age.

Gibbs (1990, 1992) showed that the source of a metaphor provides the raw material for potential metaphorical reasoning about the target. For example, a line drawn on paper is a static object, but through the LINES ARE PATHS fictive motion metaphor (Talmy, 1991), a student projects motion on to the line. Because they are familiar with the source (motion), they know that travelers move along a path. Sfard (1997) argued that students can create new target entities, in this case, virtual travelers that move along the paths. Using this source understanding, students can view static lines as paths for moving travelers. LINES ARE PATHS is an example of a *grounded* metaphor (Lakoff & Nunez, 1997) in which the source (motion through space) consists of a person's intuitions and mental simulations of simple motion experiences. Nesher's (1989) exemplification component and Post, Wachsmuth, Lesh & Behr's (1985) *embodiment* are both subsets of a metaphor source. Whereas exemplification components are systems, embodiments are much smaller pieces that students translate into mathematical symbols.

Inadequate understanding of the source of a metaphor limits a person's reasoning through that metaphor. For example, when told that ROOTS OF A NUMBER ARE COMPLEX PLANE ROTATIONS OF ONE ANOTHER, a lay person who does not understand the "complex plane rotations" source makes no metaphorical inferences at all. In contrast, a mathematician can view a root of a number as a vector in a complex plane and the other roots as rotations of that vector around the origin. The ROOTS metaphor is an example of a *linking* metaphor (Lakoff & Nunez, 1997) which projects understanding of one branch of mathematics on to another (in this case from geometry → arithmetic). Unlike grounded metaphors, a person understands the source of a linking metaphors through one or more other metaphors, possibly in coordination with other non-metaphorical relationships and algorithms. (Understanding of the complex plane includes reasoning through several metaphors such as POINTS ARE LOCATIONS, and POINTS ARE INTERSECTIONS OF LINES, and NUMBERS ARE LINES [Chiu, 1996a]). Linking metaphors typically project understandings from one branch of mathematics to another to create new insights.

In short, source experience and understanding facilitates and limits reasoning through both grounded and linking metaphors.

Differences in Prior Target Understanding

Metaphorical reasoning creates a mapping between a source and an inherently different target, so target knowledge (particularly in experts) curtails metaphorical inferences, especially incorrect ones. Lakoff and Johnson (1980) defined the target as the situation or topic that a person is trying to understand. Because a source and its target are inherently different phenomena, there must be omissions or invalid metaphorical inferences (Nolder, 1991). Expert teachers have a lot of target knowledge, so they introduce metaphors recognizing potential conflicts with standard mathematics. In contrast, novice students have little target knowledge and are less likely to recognize the limitations of the metaphors that they generate. Inappropriate metaphorical inferences may cause *misconceptions* (Clement, 1982; Davis & Vinner, 1986; Shaughnessy, 1977; Smith, diSessa, & Roschelle, 1993). Reasoning through PRIME NUMBERS ARE PRIMARY COLORS (Nolder, 1991), students may metaphorically infer that because there are a finite number of primary colors, there must be a finite number of primes. However, this metaphorical inference is false. Metaphors can contradict standard mathematics, so treating them as absolute rules rather than transitional tools for inquiry leaves one vulnerable to these potential pitfalls. As students develop expertise, they receive negative feedback on incorrect metaphorical inferences about the target, so experts are more likely to recognize invalid metaphorical inferences. Eventually, they learn that metaphorical inferences are not always reliable, so experts do not justify their results through metaphors in formal arenas such as journals (e.g., *Acta Mathematica*). Experts are more likely to avoid both invalid metaphorical inferences and metaphorical justifications.

A person's prior target (mathematics) knowledge competes with metaphorical inferences and discourages its use. In Resnick and Omanson's (1987) study, their children began with prior knowledge of the target mathematics (subtraction) that included faulty "buggy" procedures (Brown & Burton, 1978). Resnick and Omanson taught the children NUMBERS ARE GROUPS OF BLOCKS for multi-digit subtraction, but they continued to use their old, faster but faulty procedures instead of reasoning metaphorically to solve problems on the post-test. So, prior target knowledge can curtail (or completely block) metaphorical reasoning. As a result, people with competing non-metaphorical target knowledge are less likely to reason metaphorically.

Differences in Source-to-Target Projections

The one-way source-to-target projections determine specific metaphors and differ among novices and experts with respect to density and automaticity.

Lakoff and Johnson (1980) defined source-to-target projections as the specific mappings that assign the role of a specific source entity to a specific target entity. Unlike bi-directional perspectives which focus attention on common higher order relations (Lesh, Landau, & Hamilton, 1983), this view of metaphorical reasoning focuses on a unidirectional projection from a source to a target. Lakoff & Johnson (1980) argued that people understand and use phrases such as "unemployment is down again" and "prices rose 3%" (MORE IS UP metaphor), because they experience everyday correlations of heights and quantities. Piles with more papers, heaps with more leaves, stacks with more books, etc. all tend to be higher than their counterparts. (Height is not equivalent to quantity as larger bodies of water are rarely higher.) Because children understand relative height more easily (greater distance from the ground) than quantities, they may make sense of quantities by projecting their intuitive understanding of height on to it (higher → more, lower → less, rising → increasing, etc.). In contrast, people do not use UP IS MORE; they do not say "the mountain was numerous" or "the plane multiplied into the sky." Perceptual correlations may provide the foundation for simple unidirectional, grounded metaphors.

A source and a target yields different metaphors depending on their specific source-to-target projections (Gentner, 1989) or *metaphorical role castings*. Students can use splitting objects to understand either integer multiplication (Confrey, 1994) or integer division. In the case of MULTIPLICATION IS SPLITTING/DIVISION IS MERGING, quantity is the key property. For example, given four objects, splitting each object into three pieces yields twelve objects ($4 \times 3 = 12$). In contrast, size is the key property for MULTIPLICATION IS MERGING/DIVISION IS SPLITTING. Starting with lumps of clay of size four each, we can put three of them together to create a lump of clay of size 12 ($4 \times 3 = 12$). Different metaphorical role castings create multiple metaphors from a single source and a single target.

A metaphor's potential productivity depends on the number of projections from the source to the target. Intricate links may connect a metaphor's source and target as in EQUATIONS ARE BALANCED SCALES (see MacGregor [1991] for additional equation metaphors). Both sides of the equation (scale) must have the same value (weight). Replacing one expression (group of objects) with an equal expression (group of objects with the same weight) does not change the validity of the equation (balance). Students can also understand reflexivity, symmetry, and transitivity of values (weights) using this metaphor. If simply told that "equations are balanced scales" however, a novice will not generate all of the possible source-to-target projections (and metaphorical inferences) immediately, in contrast to Lakoff's (1993) fixed correspondences view for conventional metaphors. In particular, Chiu and Gutwill (1991) showed that people dynamically create different metaphorical inferences while solving a single problem. Chiu (1998) showed that graduate school students, compared to middle-school students, displayed more source-target projections and more metaphorical inferences for each arithmetic metaphor. So, novices are more likely than experts to miss appropriate source-to-target projections that would otherwise help them solve problems.

Researchers disagree over whether metaphorical reasoning serves as an ephemeral scaffold or a permanent fixture. Searle (1979) claimed that metaphorical phrases understood automatically must be *dead metaphors*, in which the person only knows the target, not the source. Likewise, Post et al. (1985) argued that abstraction during developing expertise entails target understanding without any concrete source. However, Schunn and

Dunbar (1996) have shown that scientists reason metaphorically when dealing with difficult research problems. Furthermore, Chiu (1996a) showed that graduate students do not use metaphors to solve routine arithmetic problems but can describe them when asked to do so. Vygotsky (1978, 1986) argued that experts capitalize on available resources for mediating their solution of difficult problems, resources that they do not use for simpler problems. In addition, Anderson (1987) showed that experts *compile* their *knowledge*, omitting intermediate steps (a → b → c → d becomes a → d) but retaining access to their original procedure. Using ARITHMETIC IS MOTION, a person may initially solve "–6 + 2" by walking backwards six steps from the origin (zero) to the location –6 on a path diagram and then walking forward two steps (toward the positive region) to reach –4. As that person acquires expertise, he or she omits unnecessary parts of the metaphor by: (a) starting at the location of the first operand (–6) rather than at the origin (0), (b) solving the problem without drawing a number line, etc. Eventually, the person recalls only the initial target conditions (–6 + 2) and the result (–4), bypassing metaphorical reasoning to reason only in the target. For grounded metaphors in particular, a person's familiarity with the intuitive source allows them to recreate the metaphorical projections from the source on to the target. The scant empirical evidence supports the view that metaphorical reasoning serves as a permanent resource rather than a temporary scaffold.

In short, age (beyond a low threshold) does not influence metaphorical reasoning, so a person's source understanding provides the potential for metaphor reasoning. His target understanding curtails it. His source-to-target projections specify the actual metaphorical inferences. Experts have more productive and more efficient metaphorical reasoning capabilities because of their greater source knowledge and automated source-to-target projections. They also have greater awareness of each metaphor's limitations from their greater target knowledge.

Multiple Metaphors

People can use multiple metaphors to integrate mathematical ideas and to mathematize problem situations.

Composition of Different Metaphors

Reasoning with multiple metaphors, students can integrate their mathematical understandings by connecting different target mathematical concepts through the same source and by examining the same target concept through different sources.

Overlapping Sources, Different Targets

Chiu (1996a) showed that high school students used multiple metaphors to create relationships between different target concepts from their source relationships. Because experts have greater exposure to additional metaphors, they may build more connections across different metaphors to create more coherent mathematical knowledge than novices do. For two metaphors M1 (source S1 → target T1) and M2 (S2 → T2), students can recognize source relationships between S1 and S2 and project them on to the targets to create relationships between T1 and T2. MacGregor (1991) showed that middle school students combined the following metaphors to explain different aspects of algebra: ARITHMETIC IS MANIPULATING OBJECTS, VARIABLES ARE BOXES and EQUATIONS ARE BALANCES. Students combined the MANIPULATING OBJECTS and BOXES metaphors to understand variables as boxes enclosing unseen objects. Because boxes are collections of objects in the source, students performed arithmetic operations on variables as well as on numbers. Combining these two metaphors with the BALANCES metaphor, students understood that identical actions to both sides of the scale (e.g. putting two identical boxes on to each scale pan) do not change the balance (or lack of balance) in the source. Likewise, performing the same arithmetic operation to both sides of the equation does not change the validity of the equation in the target.

Different Sources, Same Target

Chiu (1996a) showed that students used multiple metaphors with different sources to understand a single target, thereby overcoming the limitations of individual metaphors. ARITHMETIC IS MANIPULATING OBJECTS capitalizes on children's immediate recognition of small quantities (three or less) without counting (*subitizing*, Klahr & Wallace [1976]) and suffices for positive numbers. However, it does not help students understand negative numbers or understand zero as a number. (Some metaphorical extensions [such as NEGATIVE NUMBERS ARE ANTI-MATTER] are problematic because young students do not intuitively understand the source [anti-matter] well). ARITHMETIC IS MOTION covers the realm of negative numbers and treats zero as a location (origin) like all other numbers, but it does not capitalize on children's subitizing. As a result, teachers use both metaphors, introducing arithmetic through the OBJECTS metaphor first and using the MOTION metaphor later. Researchers have tried to combine both sets of metaphorical inferences into a single complex source situation. Schwarz, Kohn and Resnick (1993/94) implemented a learning system (Nesher, 1989) on a computer program that combined motion and anti-matter attributes in trains that "eat each other" to teach negative numbers successfully to three of their four children. When learning a target concept, students can use multiple metaphors with different sources to overcome the limitations of each metaphor. As a result, experts are more likely than novices to understand a particular concept through multiple metaphors and reduce their reliance on a particular metaphor.

In short, experts' greater scope of reasoning through multiple metaphors increases the coherence of their mathematical understanding while additional metaphorical perspectives on a particular concept decreases their reliance on any one metaphor.

Mathematizing Problem Situations Metaphorically

Students' target (mathematical) knowledge becomes more robust and reliable through feedback on its productive and unproductive uses as they develop expertise. So, novices who have little such feedback may return to the metaphor(s) that they used to understand the mathematics, even when applying the mathematics to a problem situation different from the metaphor source. In his study of middle school students solving a stock market problem, Chiu (1996a) showed that students metaphorically project arithmetic operations and relationships on to the financial transactions (FINANCIAL TRANSACTIONS ARE ARITHMETIC COMPUTATIONS). Furthermore, several novice students, such as JO, had difficulty with the negative number arithmetic and used ARITHMETIC IS MOTION to make sense of the arithmetic. JO used an intuitive source (motion) to understand a target (arithmetic) that he then projected on to another target (stock market), thereby creating a chain of metaphors (see Table 2).

In this view, every transfer of a learned concept or procedure to a new problem entails a metaphorical mapping from mathematics (source) to the problem situation (target), so applying an algorithm to a problem is not automatically successful. A solution relies in part on the projection of mathematical entities on to appropriate problem situation entities, possibly

Table 2. A Chain of Metaphors, Using Arithmetic to Compute Stock Market Transactions and Motion to Understand Arithmetic

Stock market (target 2)	is	Arithmetic (source 2 / target 1)	is	Motion (source 1)
Value	←	Number	←	Location
No net gain or loss at the start of today's transactions	←	Zero	←	Origin
Profit	←	Positive number	←	Location in front of the origin
Loss	←	Negative number	←	Location behind the origin
Net gain or loss	←	Sum all numbers	←	Complete journey of all motions
Determine effect of transaction A	←	Add A	←	Move A steps
Determine effect of a profit	←	Add a positive number	←	Move A steps forward
Determine effect of a loss	←	Add a negative number	←	Move A steps backward
etc.		etc.		etc.

as difficult as learning the mathematics. Facing difficult mathematical relationships in a problem situation, a novice may return to a metaphorical understanding of the mathematics, thereby creating a chain of metaphors from an intuitive source to the mathematics to the problem situation. In contrast, experts are less likely to have difficulty with the mathematical relationships and hence less likely to use chains of metaphors in this way.

Contrast with Other Types of Reasoning

To further clarify metaphorical reasoning, consider how it differs from five related types of reasoning: a) intersection reasoning, b) example-based reasoning, c) symbolic play, d) symbolic mnemonics, and e) distributed reasoning (see Table 3).

Intersection Reasoning

Researchers have argued that a person learns properties common to both the source and the target through intersection reasoning (Lesh, Landau, & Hamilton, 1983; Nolder, 1991). For example, a novice may learn about even numbers by recognizing that four dogs and six chairs can both be

Table 3. Comparison of Different Types of Reasoning

Reasoning type	Conceptual relationships	Two situations	Goal of target understanding	No need for prior target understanding	Project different properties on to target
Intersection	✔	✔	✔		
Example/ Case-based	✔	✔	✔	✔	
Symbolic play	✔	✔		✔	✔
Symbolic mnemonics		✔	✔	✔	✔
Distributed	✔				
Metaphorical	✔	✔	✔	✔	✔

arranged in two equal rows. To recognize the common higher order relations, a person must develop them in the target before intersection reasoning can occur. Therefore, intersection reasoning requires significant, prior understanding of the target. If a person knows little about the target, he or she cannot use intersection reasoning. In contrast, metaphorical reasoning unidirectionally projects source properties on to the target to create commonalities, rather than highlighting pre-existing commonalities. As discussed earlier, people reason through the metaphor MORE IS UP, but not UP IS MORE. (Analogical reasoning [Gentner, 1989; Gick & Holyoak, 1980, 1983; Holland, Holyoak, Nisbett, & Thagard, 1986] is a concatenation of separate intersection [aka "matching"] and metaphorical reasoning [aka "carry-over"] processes [Sfard, 1997].) Unlike intersection reasoning, metaphorical reasoning requires little target knowledge and projects source properties unidirectionally on to a target to create new commonalities.

Example-Based Reasoning

Reasoning through examples (Neves, 1981; Rissland, 1985) or case-based reasoning (Kolodner, 1992), also uses a prior situation like metaphorical reasoning. However, an example-based reasoner searches for a prior source that is virtually *identical* to the current problem situation, unlike metaphorical reasoning which uses an inherently different source. After mapping the appropriate entities from the source to the target, the example-based reasoner tries to repeat his actions in the prior solution. For example, consider a textbook problem with a given solution:

A train must travel 300 miles to Chicago. If it moves at 75 mph, how much time will pass before it reaches Chicago?
300 miles / 75 mph = 4 hours.

Example-based reasoning succeeds easily in word problems such as:

Ana is driving at 60 mph. How much time will Ana need to reach her aunt's home which is 180 miles away?

At a strictly computational level, the student can replace numbers. For example, 75 has "mph" next to it and so does 60, so replace 75 with 60. Similarly, replace 300 miles with 180 miles and divide (180 / 60 = 3 hours). However, problems such as the following are more difficult:

Pedro must travel 400 miles to reach his uncle's house from work, and he drives at 50 mph. If he has already driven for an hour, how much longer will it take for him to reach his uncle's house?

The example-based reasoner must adapt to the new situation to avoid simply substituting the numbers as before to obtain 400 / 50 = 8 hours. Because problem solvers using example-based reasoning try to replicate a prior solution, they must invoke alternate methods to address differences between the source and the target (Kolodner, 1992). Unlike example-based reasoning, metaphorical reasoning uses a source that is inherently different from the target.

Symbolic Play

In symbolic play (Piaget, 1962), people project objects from their imagination on to the current physical environment to explore their imaginary world (Marjanovich-Shane, 1989). For example, children playing "pirates" transform their sandbox into a ship and bail sand (water) from their sinking sandbox (ship). Although the sandbox and the sand serve as memory aids to facilitate their play, the children also played "pirates" without them. Unlike metaphorical reasoning, children engaged in symbolic play do not construct new understanding of the physical environment (Dent-Read & Szokolsky, 1993), and source properties override target properties. The children explore the source (pirate ships), not the target (sandboxes). Furthermore, children typically resolve conflicts between source properties (moving ship) and target properties (stationary sandbox) in favor of the source (ship moves). In metaphorical reasoning however, target properties override source properties in conflicts because the goal is making sense of the target. In short, a person engages in symbolic play to explore the source and reasons metaphorically to understand the target.

Symbolic Mnemonics

Symbolic mnemonics rely on the serendipity of the communication or recording media, such as talk or writing, to encode a memory aid. They are independent of the underlying conceptual meaning and depend solely on perceptual cues and transformations of the representational media. Alliteration (square's sides are the same size) and rhymes (the ships sailed two by three, just six dots in the sea) can help children remember mathematical relationships. Or, children may learn that $7 - (-2) = 7 + 2$ by imagining that one of the –'s rotates itself ninety degrees and moves on to the other "–" to form a "+." Unlike symbolic mnemonics, metaphorical reasoning does not depend on a particular notation or language.

Distributed Reasoning

Finally, distributed reasoning exploits the immediate environment directly. Gibson (1950, 1979) argued that a person or an animal's environment facilitates or *affords* particular behaviors. Moreover, these affordances are relational and dependent on the features of the person that engages in the activity. For example, a sturdy, knee-high rock affords sitting more so than a shoulder-high boulder. Several researchers (Brown, Collins & Duguid, 1989; Carraher, 1986; Lave, 1988; Scribner, 1985) emphasize the frequent utilization of their environment to solve math problems. In Lave (1988), a man obtained 3/4 of 2/3 of a cup of cottage cheese without any arithmetic computation. Using a measuring cup, he scooped out 2/3 of a cup of cottage cheese. Then he divided it into four sections with a knife and removed one of them. He simplified a potential fraction multiplication and measurement problem into a measurement and partition problem by using his environment. Unlike distributed reasoning, metaphorical reasoning invokes a different prior situation to frame the problem rather than only using resources in the current physical environment.

In short, metaphorical reasoning differs from intersection reasoning, example-based reasoning, symbolic play, symbolic mnemonics, and distributed reasoning with respect to at least one of the following properties: prior target understanding, projection of different properties on to the target, goal of target understanding, conceptual relationships, and relating two situations.

Conclusion

This article takes a step toward explicating metaphorical reasoning's origins, structures, uses, and interactions. People's innate brain structures' interpretations of perceptions, their mental simulations, their intuitions, and their schemas provide the source entities, relationships, and actions that they metaphorically project on to the target to create new target entities, relationships and actions. Students can use metaphorical reasoning to: understand new concepts, connect mathematical ideas, improve recall, understand mathematical representations, and enhance their computational environment. Beyond a low threshold, age does not determine metaphorical reasoning capacity, so a person's understanding of the source provides the potential for metaphor reasoning; his target understanding curtails it; and his source-to-target projections specify the actual metaphorical inferences. As people develop expertise, they streamline their reasoning, so that they can reason strictly within the target without metaphorically reasoning through a source. However, experts continue to reason metaphorically and their familiarity with the source allows them to recreate the metaphorical projections from the source on to the target. As a result, metaphorical reasoning may serve as a permanent resource rather than a temporary scaffold.

Experts' metaphorical reasoning differs from that of novices. Experts may have greater metaphorical reasoning capabilities because of their greater source knowledge (especially when used in linking metaphors) and their greater number of source-target projections. With greater target knowledge and awareness of each metaphor's limitations however, experts are less likely to reason metaphorically during routine problems or to justify the answers metaphorically. Experts' greater scope of reasoning through a metaphor increases the coherence of their mathematical understanding while additional metaphorical perspectives on a particular concept reduces their reliance on any one metaphor. Facing difficult mathematical relationships in a problem situation, a person may return to a metaphorical understanding of the mathematics, thereby creating a chain of metaphors from an intuitive source to mathematics to the problem situation.

By denying the view that students can simply absorb new knowledge, constructivists face the challenge of specifying the trajectory by which a student builds on prior resources to create a new piece of understanding. This trajectory requires specification of their innate resources (their bodies, especially their brains), their current understanding, and the critical processes and experiences that capitalized on their innate resources to create their current understanding. These processes and experiences may include simple perceptions, manipulation of physical artifacts, social interactions, and cultural influences among others. This article takes a small step in that direction by arguing that a seemingly simple expression such as "five minus negative four equals nine" may stem from reasoning based on a complex combination of neurologically-interpreted perceptions, intuitions, schemas, and metaphors.

Should teachers build on students' prior knowledge resources through metaphorical reasoning to help them learn mathematics? Students already encounter many metaphors in mathematics textbooks (Chiu, 1992) and hence, in classrooms (Mayer, 1986). However, many of these metaphors are isolated to a specific procedure, rather than systematically connecting related mathematical ideas (Chiu, 1992). A few small-sample empirical studies (Silva & Moses, 1990; Schwarz et al., 1993/94) show the successful use of metaphorical reasoning for teaching mathematics, so the issue requires more in-depth studies. Nevertheless, the existing research suggests a few heuristics. When choosing metaphors, curriculum designers should maximize both breadth and depth of coverage while minimizing invalid metaphorical inferences. Because the source and target of a metaphor are inherently different however, there are always omissions or invalid metaphorical inferences. Teachers can help students recognize these differences and avoid incorrect metaphorical inferences. As exemplified by ARITHMETIC IS MOTION, reasoning through a single metaphor can connect many related mathematical topics in detail. Because students must build mathematical understanding on the foundation of their prior knowledge, they may reason through grounded metaphors that rely on intuitive sources more easily. Virtually universal experiences such as eating and motion are likely candidate sources, but educators may also capitalize on common cultural experiences.

Because reasoning through a novel metaphor may include a complex coordination of multiple potential inferences, students can benefit from teacher guidance on appropriate uses of metaphorical reasoning. Firstly, the teacher must decide how to introduce a particular metaphor (e.g. posing a problem in an environment in which students are likely to understand the metaphor source). Secondly, students can often construct different metaphors from a single source and target. So, the teacher must focus the students' attention on important aspects and encourage them to project appropriate source properties on to target entities for interpreting mathematical representations, algorithms, conceptual relationships, etc. Because the source and the target of a metaphor inherently differ, the teacher must help students identify invalid metaphorical inferences. In addition to choosing an appropriate metaphor source for the target mathematics, the teacher plays an important metacognitive role by helping students negotiate their way through the benefits and the pitfalls of metaphorical reasoning.

Metaphors have a great deal of potential, but their successful implementation requires further evidence of their efficacy and greater understanding of their strengths and limitations. In particular, how are a person's non-mathematical resources coordinated to create a source for a metaphor? How do students learn to reason through particular metaphors? What roles should the teacher play? How do students learn to combine metaphors appropriately? By answering these questions, educators can help students learn by building on what they already know.

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Appendix A

List of Selected Brain Structures, Mental Simulations, Intuitions, and Schemas

Brain Structures

Motion-Detectors. When two different sensory neurons detect each light flash, both send signals to a motion detector. If the sensory neuron detecting the first light flash has a longer pathway to the motion detector than the other neuron, then both signals arrive at the same time to activate the motion detector. Activation of different pairs of neurons activate different detectors of motion in different directions. As result, humans also detect direction of motion along with the motion itself. See Barlow & Levick, 1965.

Edge-Detectors. To simplify a bit, there are on-off cells that detect light. When a row of cells are activated (on) next to a row of cells that remain inactivated (off), humans perceive an edge. Groups of cells must be activated as a whole, so that edge detectors are specific to both position and orientation. See Hubel & Wiesel, 1962.

Color-Detectors. Through cones and rods, humans detect light and darkness, red and green, blue and yellow. Kay and McDaniel (1978) argue that detection of other colors result from combinations of different activation levels of these color detectors. See also DeValois, Abramov, and Jacobs, 1966, DeValois and Jacobs, 1968, Marks, Dobelle, & MacNichol, 1964, and Wald, 1959.

Intuitions

Path. A stimulus (or a configuration of stimuli) that appears in one location and then in a different location a short time later has moved (or was moved) from the first location to the second. Over time, an infant may recognize that more complex stimuli such as objects will appear different, especially with further movement as a result of changes in lighting and visual perspective.

Object. Contiguous edges and color(s) that move together (or remain stationary together, This intuition involves edge-detectors, color detectors, and the path intuition. As with the path intuition, the object's appearance changes due to different lighting and visual perspectives.

Length. Stimulation of contiguous edge detectors with the same orientation.

Verticality. Spatial distance away from a grounded horizontal reference frame. Note that verticality requires judgment relative to a location, not simply identifying a location.

Container. Configuration of edges that constrain motion of an object in its interior. An object inside a container activates a subset of its interior edge-detectors (object).

Mental Simulations

Translation. Humans can mentally move an object from one location to another, maintaining the orientations of the object's edges. Translation involves edges, motion, object(s) and path(s).

Rotation. The center of an object remains in the same position while all other parts move. The orientations of all of the objects' edges change. Rotation involves edges, motion, and object(s).

Composition. Placing one or more mental objects on top of another by translation of two objects to the same general location. Composition involves edges, motion, and objects.

Scaling. (Magnification or Shrinking) Objects appear larger as they move closer to the viewer (or as the viewer moves closer to the object), activating more edge-detectors away from the viewer's visual center. Likewise, objects moving away activate fewer edge-detectors away from the viewer's visual center.

Multiplex <–> *Mass*. During extreme movement of objects away from the viewer (or viewer movement away from the objects), the objects' separate edges all appear contiguous. Likewise, as the distance between a viewer and the viewed objects decreases, an apparent single object (all contiguous edges) becomes distinct objects (some non-contiguous edges, Note that this is true only if the width of an object is greater than its distance to another object.

Schemas

Containers and Locations of Objects. For closed containers, an object must be either inside or outside a container (as indicated by the status of the interior edge-detectors of the activated contiguous edge-detectors). For open containers, the object can also be partially inside and partially outside.

Balance. Upon placing objects on the right and left ends of a horizontal beam (supported in the center by a fulcrum), the beam can either: (a) remain perfectly horizontal, (b) tilt to the left, or (c) tilt to the right (as indicated by the beam-activated edge-detectors' orientation).

Motion Along a Path. A traveler can move forward or backward along a path any distance. If location B is between location A and location C, a traveler at location A must pass through location B to reach location C.

Combining Objects. Some objects (e.g. pieces of clay) can be combined to create fewer larger objects (as indicated by different configurations of activated, contiguous edge-detectors). Given X objects and $0 < V < W = X$, combining every V objects together results in more but smaller object(s) than combining every W objects. Given 24 objects, combining every 3 objects together yields 8 (size 3) objects, whereas combining every 4 objects yields 6 (size 4) objects.

Splitting Objects. One or more objects can be cut into many smaller objects (as indicated by different configurations of activated, contiguous edge-detectors). Given X objects and $0 < J < K$, splitting each object into J pieces yields fewer larger pieces than splitting them into K pieces. For example, splitting each of 7 objects into 2 pieces yield 14 (half-sized) pieces while splitting them into 3 pieces yields 21 (one-third) sized pieces.

Appendix B

List of Selected Mathematical Metaphors

Due to space considerations, I included only the central elements of each metaphor. Many additional (or fewer) metaphorical entailments are possible for any individual. For additional metaphors, see Lakoff, Espenson, & Goldberg, 1991, and English, 1997.

Arithmetic Is Manipulating Objects

Zero ← No objects
Unit (one) ← An object
Addition, $A + B$ ← Putting B objects into the current pile of A objects
Subtraction, $A - B$ ← Removing B objects from the current pile of A objects
Multiplication, $A \times B$ ← Replacing the original A pieces with B replications of the A pieces

or Multiplication, $A \times B$ (Confrey, 1994) ← Cutting each of the current A objects into B pieces

Division, A / B ← Putting the A pieces evenly into B boxes and counting the pieces in each box

or Division, A / B ← Merging every B of the A objects into new objects

Numbers Are Groups of Blocks

Each power of ten (1, 10, 100 …) ← an object of different size

Each power of ten is equal to ten of the next lesser power of ten ← An object is equal in size to ten of the next smaller size object

Each digit of a number ← A group of blocks of the same size

A number consists of digits ← A collection of blocks consists of groups of different size blocks

Arithmetic Is Motion Along a Path

0 ← Origin/starting point

Number ← Location relative to the origin

Absolute value $|A|$ ← Distance from the origin (A steps away)

$A > B$ ← Location A is to the right of location B

Add A ← Move A steps (if $A=0$, then hop in place)

$A + B = B + A$ ← Switching the order of steps taken results in the same destination

Subtract B ¨ Turn around and move B steps

$A + (-B) = A - B$ ← Walking backwards is the same as turning around and walking forward

Addition is the inverse of subtraction ← Moving forward is the opposite of moving backwards

Additive inverse ← Direction and steps needed to return to the origin

Multiply $M \times N$ ← Take M steps, each of size N. If M is negative, turn around first. If N is negative, take steps backward.

$M \times N = N \times M$ ← Switching the number of steps and the step size yields the same destination.

M / N, e.g. $-20 / 2 = -10$ ← How many steps of size 2 do you take to go to location –20? Turn around (–) and take 10 steps, –10

$M / 0$, e.g. $4 / 0 =?$ ← How many steps of size 0 do you take to go to 4? Impossible

0 / 0 = ? ← How many steps of size 0 do you take to go to 0? Any number of steps …0,1,2…

Division is the inverse of multiplication ← Asking how to get there is the opposite of asking where are we going?

Equations Are Balances

Number ← Weight (or lift) of an object (e.g. marble or balloon)

The equation is true ← The scale is balanced, and both pans have the same net weight

Operating on two (or more) operands on one side of the equation ← Replacing two (or more) objects with one objects with the resultant/equivalent weight/lift

Operating on both sides of an equation ← Changing the weights of both pans in the same way

Two true equations can be added or subtracted to generate another true equation ← Combining the pans of two balanced scales (or removing the objects of one balance's pans from the other) creates another balanced scale.

Given a true equation, the expression on one side can be substituted with the expression on the other side in any other new equation without affecting the validity of the new equation. ← Given a balanced scale, the objects on one pan can be replaced with objects on the other pan in any new scale without affecting the balance (or imbalance) of the new scale.

Lines Are Paths

Static line ← Trajectory of a moving object

Polygon ← Path in which a traveler begins and ends at the same location

Points Are Locations

Line ← Edge dilated infinitely in both directions

X-axis ← Reference dilated edge (horizontal)

Y-axis ← Reference dilated edge (vertical) perpendicular to above reference dilated edge

Points Are Intersections

Cartesian Point ← Location at the intersection of two non-parallel dilated edges
Origin at x & y-axes ← Reference location formed by the intersection of the two reference edges
Set of all Cartesian points span the plane ← Locations at all intersections of dilated edges covers the two dimensional space

Numbers Are Lines

Ordered pair (0,0) ← Reference location designated by intersection of the two reference lines.
Cartesian function ← Collection of locations
A point uniquely determines a coordinate pair ← A location uniquely determines two edges
A coordinate pair uniquely determines a point ← A pair of edges uniquely determine a location

Linear Scales Are Paths

Number ← A location on the path
Zero ← Origin
Number N ← Location on the path N units distant from the origin
Number N ← Line that intersects the path at a location N units distant from the origin
Number a is greater than number b ← Location a is further from the origin than location b

Prime Numbers Are Primary colors

Natural Numbers ← Colors
Composite numbers ← Composite colors
One ← Transparent
Product of two different prime numbers yields a composite number ← Composition of any two different primary colors yields a secondary color
Product of two identical prime numbers yields a composite number <-X- Composition of two identical colors yields the same color
Infinite number of prime numbers <-X- Finite number of primary colors

Roots of Numbers Are Complex Plane Rotations of One Another

Number ← Vector in complex plane starting at the origin
Nth root of a number ← Rotation of the re-scaled vector around origin
Remaining N – 1 roots ← Current vector + rotation (i/N × 360 degrees), i = 1, 2, … N – 1

Sets Are Containers

Members ← Objects
x is a member of set S, $x \in S$ ← Object x is inside container S
Intersection $A \cap B$ ← Container of objects inside both containers
Union $A \cup B$ ← Container of objects inside either container
Negation ~A ← Container of all objects outside container A
Implication $A \rightarrow B$ ← The objects inside container B are inside container A
$A \vee B$ ← Object is inside container A or container B or both containers
$A \wedge B$ ← Object is inside both container A and container B
~ A ← Object is outside the container A
$A \rightarrow B$ implies $\sim B \rightarrow \sim A$ ← Container A inside container B implies both that an object inside container A must be inside container B and an object outside container B must be outside container A.