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Teaching Linear Algebra: Conceptual and Procedural Learning in Linear Transformation

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Linear algebra is generally considered by students as being abstract and boring because of its irrelevance to daily life, and it caused a negative attitude and a general failure among the students. In line with this point of view, the aim of this study is to define the reasons why students fail in learning linear algebra. In the research, 50 students who failed in three semesters consecutively have been observed for three years among a total of 280 students who took Linear Algebra II lesson in the Department of Primary School Mathematics Teaching, Kazım Karabekir Education Faculty, Atatürk University. In this course, data has been collected from answers given by the students to questions asked in the exams and from interviews made with the students. Based on the assessment of the students' exam papers and the

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results obtained from the conducted interviews, it has been observed that students do not have difficulty in learning the concepts, definitions and formulas related to linear algebra and operational information, but they experience difficulties in the implementations of the definitions and the concepts that they have learned.

Key words: linear algebra, conceptual knowledge, procedural knowledge

Introduction

It is well known that every branch of science has its own specific methods of teaching in accordance with its purposes. Teaching methods in line with the structure of mathematics should achieve the following purposes of helping students to learn (Van de Walla, 1989):
1. conceptual knowledge of mathematics,
2. procedural knowledge of mathematics, and
3. connections between conceptual and procedural knowledge.

These purposes are defined as relational conception. The relational conception may be explained as understanding operational concepts (related concepts and their structures) in mathematics, expressing them with symbols, and benefiting from their facilities; conceiving the techniques of procedures in mathematics, and stating them with symbols; and getting an understanding of relational connection among methods, symbols and concepts.

By studying students' knowledge of mathematics in terms of learning psychology, two kinds of mathematical knowledge are noted. The first one deals with entirely mechanical data including abilities such as recognizing the symbols, doing the operations. The second one is the ability to put symbols into mathematical concepts, forming relationships among them and doing operations by using them (Baki, 1998). While in operational data, it is adequate that one knows only the operational process without realizing the conceptual relationship among them, it is important to know conceptions in conceptual data (Baki, 1997). The operational approach considers mathematics as a set of presented data which can be directly introduced to students. On the other hand, the conceptual approach of mathematics opposes to this view in that students can learn mathematics only with their own efforts (Noss & Baki, 1996). As Hiebert and Levefre (1986) suggests, operational knowledge is both the symbolic language of mathematics as well as the data of operations and formulas for solving the related problems, and they define conceptual knowledge as a network which includes the special parts of operational data in it and as the connections obtained from these specific parts of information. The conceptual and the operational data is a kind of component with two relativities and both of them are important for mathematics (Hiebert & Carpenter, 1992). It is possible to attain a permanent capability of mathematics with having only a balanced knowledge of conceptual and operational mathematics (Baki, 1998).

Although conceptual learning should be dominate in mathematics teaching, operational learning was mostly focused on. Thus, a balance could not be achieved between operational and conceptual learning in mathematics. In the absence of such balance, the subjects could not be learned at the conceptual level (İşleyen & Işık, 2003).

Teachers consider the capability of mathematics as using formulas, operations and rules properly, and they believe that it is adequate for students if they are able to calculate and use formulas and come to the accurate conclusion of a question. However, it is not adequate for a mathematics learner to have only the capability of calculation and knowledge of specific formulas, it is more important that the person should be able to have conceptual understanding of mathematics and to undergo a process of improving this understanding as well (Baki, 1996).

According to the results of Oaks' (1990) research, mathematics subjects are memorized instead of being learned due to the fact that mathematics lessons are not taught with a conceptual emphasis. The result of this research shows that most students are not aware of what is meant by the mathematical concepts when using the operations. They believe that learning mathematics means to perform mathematical operations on the symbols which have no meaning, and they try to learn mathematics by means of memorizing.

An observation of Bostwana on teachers in classroom teaching reveals that their teaching method is memorization of formulas and algorithm during courses in the classrooms. Many students graduated from schools without having the basic capability of reading, writing and counting (Mapolelo, 1999). The main reason why students attempt to learn mathematics by memorizing formulas and rules is that there is a great difficulty in both teaching and learning mathematics because of its abstractness. This failure in understanding leads to prejudice and a rather negative attitude towards learning mathematics, and learners tend to memorize formulas and rules without focusing on the concepts behind them (Konyalıoğlu, 2003). To have accurate and proper comprehension of a mathematical concept, one should give a meaning to it in mind and know how to interpret it (Umay, 1996).

Kennedy and Tipps (1991) stated that mathematics being an abstract lesson was one of the complaints among the students, but when teaching was conducted by means of attaching importance to mathematical concepts, working with concrete models and providing an environment for discussions, the number of student complaints decreased and mathematics lessons gained considerable success. This has been one of the measures taken in order to prevent the development of negative attitude against mathematics (Altun 1995). It is necessary that the concepts of mathematics should be given clearly and properly, the relations should be put out in a clear and direct way, and all the expressions apart from these subjects should be completely defined. Students should learn the meaning of the formulas, concepts and rules without memorizing them, and only in this way they will be able to comprehend mathematics and acquire an adequate capacity of it (Soylu, 2005).

In mathematics lessons, teaching is also conducted without taking into account the social lives and the previous mathematics knowledge of the students, which is also a reason why students cannot learn mathematics successfully. In order to eliminate this failure, lecturing should be given in a way that the previous mathematics knowledge of the students and their social lives are taken into account, so that students can participate actively in the lesson, and are convinced about the subject given. In other words, the students must be able to envisage the concepts that are related to the given subject (Fleener, Westbrook, & Roger, 1995). As stated by Altun (1995), importance must be attached to the concept when teaching mathematics. For instance, a conceptual confusion will arise if the linear transformations be given before explaining the concepts about transformations clearly.

Stating that great importance should be attached to the concept when teaching mathematics, Yıldız (2000) lists the reasons as follows:

- The recent teaching approaches accept that conceptual teaching has a great significance for permanent learning.
- It can be admitted that if a student is able to apply his/her earlier knowledge to a new situation, he/she has an accurate learning.
- The knowledge attained from daily life and earlier experiences has a significant effect on learning new knowledge.
- If a student has misperceptions about some specific topics, it is most likely that this wrong perception will mislead the learner.
- New information is discovered every day, resulting from development of science and scientific discoveries. This process is so rapid that it extends the ability of perception. In order to practice this new entity, teaching methods should be conceptual rather than operational.
- It is not possible to give conceptual teaching without correcting the wrong information students learned from their earlier education and from their interaction with the environment.
- In conceptual learning, the teaching process should be hierarchical, from simple to complex subjects.

Harel (1989) and Wang (1989) considered that students have difficulty in learning concepts in linear algebra, on the other hand, they can easily handle the accounting and calculating parts in their research. Students get difficulty with linear algebra because there are too many definitions in linear algebra, and these definitions are not practiced during the lesson. The students said that when it comes to practice these learned definitions, they feel as if they have gone to a new planet and they got lost. When they are asked to do exercises and practices about linear algebra, most of the students get very confused and they cannot build up the necessary connections with the concepts (Dorier, 1998).

Several researches have revealed that many students have algorithmic ability in linear algebra. Carlson (1993) explained that students do not make mistakes on simple algorithmic calculating as multiplications of certain matrixes, giving definitions and the solution of linear equation systems. However, this is not the case when it comes to more abstract concepts such as linear independency, space, sub-space and linear transformations (Sabella & Redish, 1995). This situation reveals that conceptual learning is not completely accomplished in linear algebra.

Carlson (1993) summarizes why it is so difficult to teach and learn certain subjects of linear algebra in a couple of items:

- The students of linear algebra have no precondition data, and this situation is not taken into account by the teachers.
- The main problem is not the learning of algorithmic calculations; it is the learning of the concepts about the given subjects. Nevertheless, many students primarily the Americans have almost no information and skills except for mathematical calculations.
- The appropriate teaching methods and multidimensional practices are not carried out in order to teach the concepts about the subjects.
- In linear algebra lessons, implementations which require students to make analysis and interpretation in relation to the lessons are not used.
- The students have not the opportunity to use the concepts they learned, and they cannot connect the new data meaningfully with those of the earlier (Sabella & Redish, 1995).

The reason why students can not comprehend the basic notations is that abstract concepts are introduced to them without constructing the basic conceptual information firmly (Harel, 1989; Wang, 1989). Harel (1989) adds that graduate students experienced such problems as well.

So there occurs confusion when various definitions from different stages are used randomly, and that is the reason why students fail in learning linear algebra. In order to eliminate these difficulties, a dynamic geometrical programming should be used, and an elaborate way of teaching should be followed starting from eigen values, eigen vectors, and the geometric views of linear transformations and vectors (Dreyfus, Hillel, & Sierpinska, 1998).

Linear algebra is given in the first or second year at the universities. It is well known that students consider linear algebra as a very difficult subject. Linear algebra is seen as being irrelevant to other subjects of mathematics and being too abstract. As a consequence, in order to teach conceptual and operational subjects together, the geometric subjects and concepts should be given in a meaningful way (Gueudet-Chartier, 2003).

As being a basic branch of modern algebra, linear algebra is a subject affecting all branches of mathematics. For being applicable to all physical branches of science, linear algebra is indispensable for all the college students. Linear algebra can be useful and necessary in many departments such as electric-electronic engineering, chemical engineering, economy, statistics, social sciences and biology. Being one of the most significant subjects of linear algebra, linear transformation has a significant role on physics, social sciences and economy, as well as on many parts of mathematics (Işık, 2000). Since linear algebra has a very significant role on many occasions, the importance of conceptual teaching in linear algebra, and the difficulties of the teaching process have been studied.

Method

Exemplification of the Study

In this research, 50 students were chosen from a total of 280 students studying primary mathematics education in Kazım Karabekir Education Faculty, Atatürk University.

The reason for having included these 50 students in the sample is to research, in terms of operational and conceptual learning, the reasons for the failure of the students who failed in Linear Algebra II in three semesters consecutively. The research was carried out from the second term of academic year 2002–2003 to the second term of academic year 2004–2005.

Aim of Study

The aim of this research is to find out the reasons for the failure of the 50 students who studied in the Department of Primary School Mathematics Teaching, Atatürk University. The chosen students failed in Linear Algebra II for three times successively. This study also aims to research whether failure to balance the operational and conceptual teaching in linear algebra is among the reasons or not.

Procedure and Data Analysis

In this study, the reasons for the failure of the students studying primary mathematics education have been researched. For that purpose, 50 students who failed the subject were selected randomly among the students that took the Algebra II Lesson in the semesters 2002–2003, 2003–2004 and 2004–2005, and they have been observed for 3 years in this period. Cooperation has been made with the academician who gave this lesson in those semesters. Cooperation has been made with the academician only in the stage of preparation of visa and final questions.

The methods and the techniques used by the academician in lecturing have been entirely left to the academician's discretion. The information required from the students in the examination questions is divided into three categories:

1. Definitions related to Linear Algebra
2. Operations related to Linear Algebra
3. Making the application of the definitions related to Linear Algebra

While determining whether the prepared questions match the information that we intend to research or not, in other words, while determining the validity and the reliability of the questions, opinions of four academicians who are specialized in this field have been taken. The questions which are to be asked to the students in the examinations have taken their final form in accordance with the opinions of these academicians.

In order to find out the reasons for the students' failure in Linear Algebra lesson, information collection has been conducted in two ways:

1. Examining the frequencies of the answers given in the exam papers of the students and calculating them.
2. Making interviews with the students who have been selected randomly.

In the research, the reason for examining the answer papers of the students in visa and final examinations is that the students try anxiously to get good grades and write everything they know in the examinations. Due to the fact that the students do not take the research oriented surveys very seriously, obtaining the required information has become much more difficult

Findings

This section covers the data obtained from the visa and final examinations of the students who took Linear Algebra II lesson in three semesters consecutively, and the answers given by the students to the questions and the interviews made with the students. The answers of the students are classified in a few groups not independent of each other. In other words, it is possible that the answer of a student may take place in more than one group. The classification is divided into five categories: correctly defining the subject, incorrectly defining the subject, leaving the subject unanswered, managing the implementation of the subject, and failure to conduct the implementation of the subject. Each of the results is given in letters in order to compare and contrast the frequencies of the answers, and all the results are stated as below.

Question 1

This question which requires the ability of defining linear transformation, detecting elements from defining set, operating accurately with the help of these detected elements (vectors) has been asked to the students during their first and third attendance in the course of Linear Algebra II:

"*Define linear transformation, and find out if that function of* $T : R^3 \to R^3$, $T(x, y, z) = (x, y, 1)$ *is a linear transformation.*"

The responses of the students during the first term are stated below:

a. 40 students defined linear transformation properly.
b. 5 students failed to define linear transformation.
c. 5 students did not answer the question.
d. 15 students, who defined linear transformation accurately, also used the definition properly and gave correct answers to the question.
e. 25 students of those who defined properly, failed in using it.

The answers of the students taking the course of Linear Algebra II for three times are stated below:

a. 43 students defined linear transformation properly.
b. 6 of them failed to define it.
c. 1 of them did not answer the question.
d. 19 students of those giving correct answers also managed to use the definitions properly and gave correct answers to the question.
e. 24 students of those having done the definition properly, failed to use the definition accurately.

The frequency disperses of the students' answers are exhibited in Table 1. The answer of a student to this question is shown in Figure 1.

Table 1 Frequency Disperses of Students' Responses to Question 1

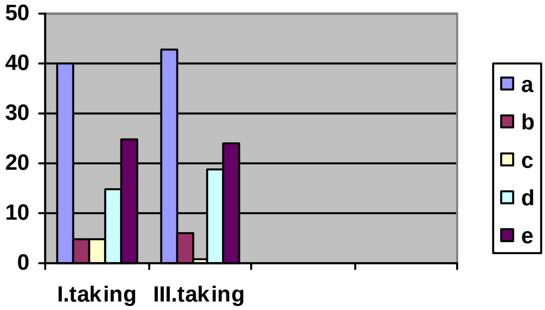
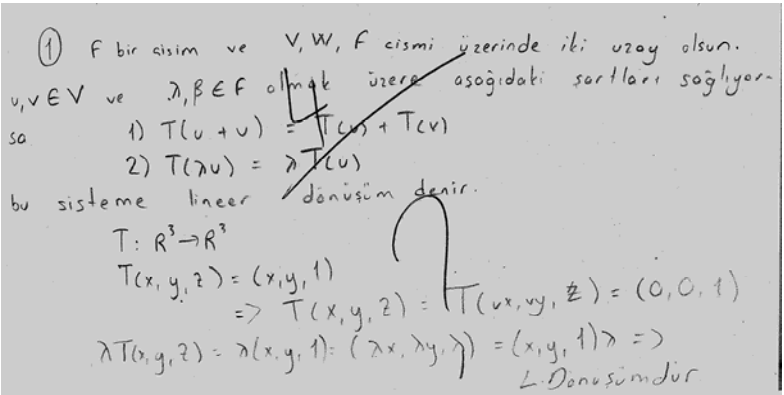


Figure 1 Sample of the Exam Paper of Student A on Question of Use of Definitions and Practice of Linear Transformations



The interview with Student A on this question is reported below:

Interviewer: It is clear from your test results that although you defined correctly the linear transformation, you failed to state if the given function is a linear transformation or not. What can be the reason for this conclusion in your opinion?

Student A: I memorize the definitions without knowing what their meanings are, so I have no difficulty in definitions. If I encounter an example of the questions about practicing definitions, I answer the question; otherwise I fail in answering to them. I think I have trouble at practicing because I only do memorization.

Interviewer: You have been taking this course for three terms. What is the difference of linear transformation from a function you learned before?

Student A: I know the linear transformation only as a definition. To put it in another way, suppose that F is the substance, and V, W are two linear spaces on F. As put in this way: $\lambda, \mu \in F$ and $u, v \in V$ if the transformation of $T : V \rightarrow W$ gives out those conclusions:

i. $T(u+v) = T(u) + T(v)$

ii. $T(\lambda u) = \lambda T(u)$, T is a linear transformation. I know this definition. But I do not know exactly what the relationship of this definition with a function is and what the similar and different points are.

Question 2

This question which requires such skills as having learnt the definition of nucleus, using the definition of nucleus in answering the question, selecting the elements of nucleus among various elements in the defining set has been asked to students during their first and second taking of the course, and their answers in the two terms are stated below.

"*Define the nucleus of a linear transformation, and find out if the linear transformation of*
$T: R^3 \rightarrow R^3,\ T(x,y,z) = (x+y-2z,\, 2x-y+z,\, y-5z)$
is a base and dimension."

The responses of the students during the first term are stated below:

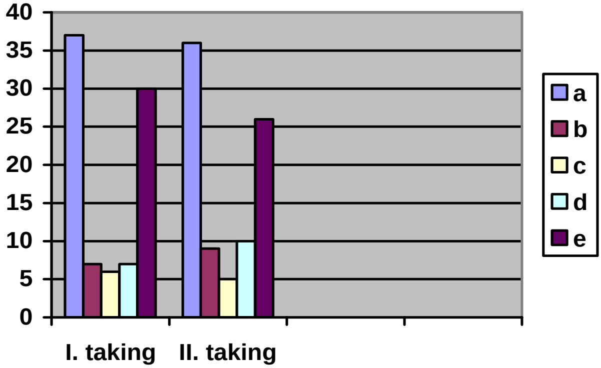
a. 37 students defined the kernel of a linear transformation correctly.
b. 7 students failed to answer the question correctly.
c. 6 students did not answer the question.
d. 7 students who answered the question correctly also put the definition into practice properly.
e. 30 students who defined the question correctly failed to put the definition into practice.

The answers of the students after taking the Linear Algebra II the second time are stated below:

a. 36 students defined the nucleus of a linear transformation correctly.
b. 9 students failed in defining the transformation.
c. 5 students did not answer the question.
d. 10 students who answered the question correctly also put the definition into practice properly.
e. 26 students failed when they put the definition into practice.

The frequency disperses of the students' answers are exhibited in Table 2. The answer of a student to this question is shown in Figure 2.

Table 2 Frequency Disperses of Students' Responses to Question 2



The interview with Student B on this question is reported below:

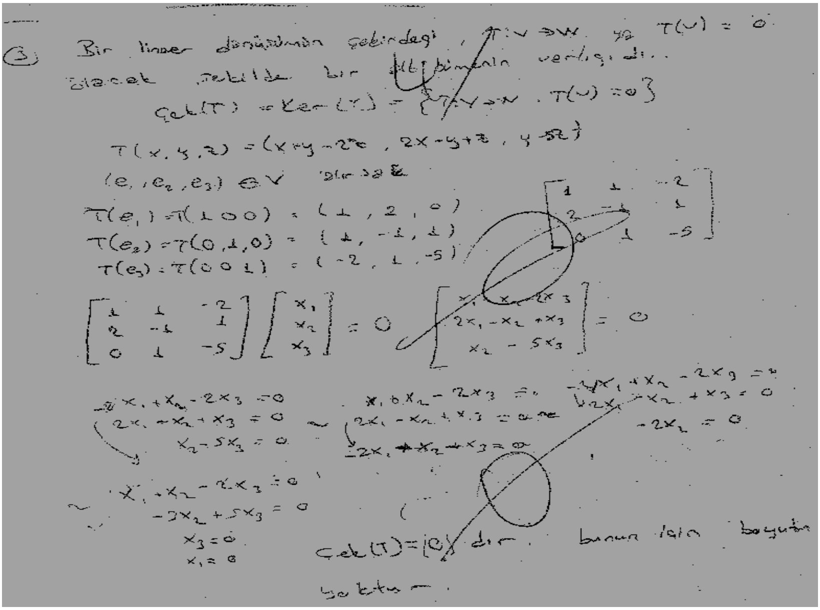
Interviewer: It is apparent in this exam paper that although you define the nucleus of a linear transformation correctly, you failed to determine the base and dimension of a nucleus. What is the reason for this in your opinion?

Student B: I could in no way comprehend the subjects of base and dimension. I can not construct them in my mind in any way. I think either these subjects are too abstract, or memorization is a wrong method. I do not know the reason exactly.

The answers given by the students for the above two questions during their first and latter time of taking the Linear Algebra II lesson show similarities with only slight differences. In view of these results, it is seen that without perceiving the meaning of the definitions or learning the definitions on a conceptual level, one can not be successful no matter how many times they take the lesson. This result is supported by the student interviews and the answers provided on exam papers by the students.

In the first and second questions listed above, it is seen that the difficulties experienced by the students lie neither in the definitions of the linear transformation nor in the kernel of the linear transformation, but in the implementation of these points. That is to say, it is observed that the students do not have difficulty in operational information, whereas they have difficulty in conceptual information.

Figure 2 Sample of the Exam Paper of Student B on Question of Use of the Nucleus of Linear Transformation and its Base



Question 3

Students who took Linear Algebra II lesson during their first and third time have been asked the question *"Define the determinant of a matrix and write the properties of the determinant"*, which requires knowing the definition and properties of the determinant of a matrix, in other words, which does not require the knowledge of implementation.

The answers given by the students in their first time of taking the lesson are as follows.

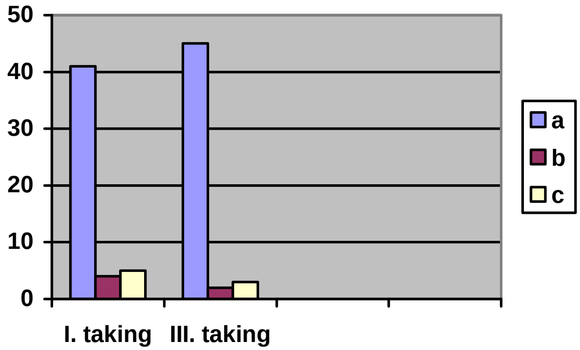
a. 41 students wrote the determinant and the properties of a matrix correctly.
b. 4 students wrote the determinant and properties of a matrix incorrectly.
c. 5 students gave no answer to the question

The answers of the students after taking Linear Algebra II lesson the third time are stated below.

a. 45 students wrote the determinant and the properties of a matrix correctly.
b. 2 students wrote the determinant and properties of a matrix incorrectly.
c. 3 students gave no answer to the question

The frequency disperses of the answers given by the students is indicated in Table 3.

Table 3 Frequency Disperses of Students' Responses to Question 3



Viewing Table 3, it is seen that the students have no difficulty on this question. What sets this question apart from question 1 and question 2 is that only the meaning of definitions is required in this question while the application of these definitions is not required. However, in questions 1 and 2, there are both the definition and the application of these definitions. This means that the information required in question 3 is completely operational information, whereas the information required in question 1 and question 2 are both operational and conceptual information. This can explain why the rates of correct answers are higher in that question for it requires only memorization of the operations, not analyzing of practicing skills.

Question 4

This question requiring the use of related definitions and practices of linear transformation has been asked to students during their first, second, and third taking of the course. The question was that:

"$T: R^3 \rightarrow R^2$, $T(x, y, z) = (x + 2y, -x + y)$ be one base of the linear transformation's definition set, R^3 as one base of the standard bases and value set. Examine if $\alpha = \{(1, -1), (1, 2)\}$ equals to $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$."

The results given by the students to the question during their first taking of the lesson are stated below:

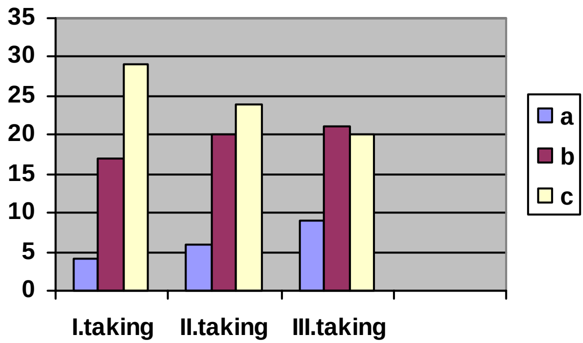
a. 4 students gave the answers correctly.
b. 17 students answered the question incorrectly.
c. 29 students gave no answer to the question.

The answers from their second taking of the lesson are stated below.

a. 6 students answer the question correctly.
b. 20 students answered incorrectly.
c. 24 students gave no answer to the question.

The frequency disperses of the answers given by the students is indicated in Table 4. The answer of a student to this question is shown in Figure 3.

Table 4 Frequency Disperses of Students' Responses to Question 4



The difference of this question from the other questions is that operational information is not required. That is to say, with the use of certain definitions and concepts, the verity of the equation $\left[T(v)\right]_{\beta} = \left[T\right]_{\alpha}^{\beta} \left[v\right]_{\alpha}$ is requested to be shown. Here is the application and analysis of the definitions or concepts. In other words, conceptual information is required in order to answer this question correctly. Therefore, it has been noted that students have difficulty on this question. The failure of the students on this question at their each taking of the lesson is due to the fact that the question requires analyzing and practicing of some definitions about linear transformations. This is also clearly shown in the answer paper of Student C below. Many operations have been made by the student on the paper; however, it is not certain how, why and for what purposes they have been made.

The interview with Student C on this question are reported below:

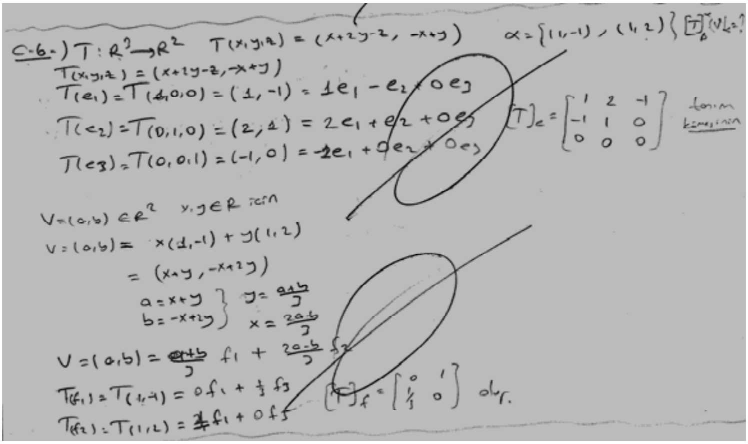
Interviewer: It is obvious that you have much information on this question, but you got confused on solving it. What is the reason in your opinion?

Student C: I could easily solve that type of questions before taking the exam. But I completely got confused when the data and the values changed in the question.

Interviewer: Can we say that you got confused because of having memorized these type of questions before entering the exam?

Student: It might be so. Because I got trouble when the items and values changed in the question. However, I was able to solve the questions while I studied before exam, and so I suppose that I have comprehended the lesson.

Figure 3 Sample of the Exam Paper of Student C on the Question of Use of Definitions and Concepts of Linear Transformations



Question 5

This question which requires the cognition that the multiplication of an element with its inverse equates to the unit element and algorithmic operations has been asked to the students during their first and third taking of the course of Linear Algebra II. The question was that:

“$\begin{bmatrix} -3 & 2 & 1 \\ -1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ *find this matrix’s inverse by using unit matrix.”*

The responses of the students during their first taking of Linear Algebra II are stated below:

a. 45 students indicated correctly that the multiplication of a matrix with its inverse equates to unit matrix.
b. 3 students failed to state correctly that the multiplication of a matrix with its inverse equates to unit matrix.
c. 2 students gave no answer to the question.
d. 44 of the students, who indicated that the multiplication of a matrix with its inverse equates to unit matrix, have also used the algorithmic operations correctly.
e. 5 of the students, who indicated correctly that the multiplication of a matrix with its inverse equates to unit matrix, failed to use the algorithmic operations correctly.

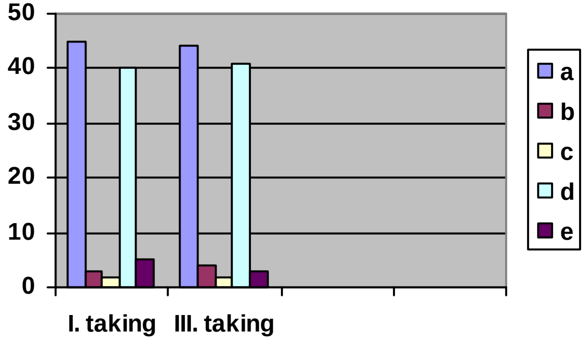
The responses of the students during their third taking of Linear Algebra II are stated below:

a. 44 students indicated that the multiplication of a matrix with its inverse equates to unit matrix.
b. 4 students failed to state correctly that the multiplication of a matrix with its inverse equates to unit matrix.
c. 2 students gave no answer to the question.
d. 41 of the students, who indicated that the multiplication of a matrix with its inverse equates to unit matrix, have also used the algorithmic operations correctly.
e. 3 of the students, who correctly indicated that the multiplication of a matrix with its inverse equates to unit matrix, failed to use the algorithmic operations correctly.

The frequency disperses of the answers given by the students to that question is indicated in Table 5.

It is seen that the ratio of correct answers in this question in which only definition and properties are asked is high in both the first and third time of taking the lesson. The performance of the students recorded in Table 1, Table 2, Table 3, Table 4 and Table 5 showed that they do not experience problems in definitions.

Table 5 Frequency Disperses of Students' Responses to Question 5



The interview with Student D concerning this question is reported below.

Interviewer: You have given the correct answer to the question. Similarly, a great majority of your friends also have given the correct answer to that question. You said that you memorized the earlier definitions without knowing the meanings since they are abstract. Aren't the concepts abstract in that question as well?

Student D: I know what the unit and inverse element mean and express it in the question. After having this cognition, we get no trouble about doing operations. We do not have problem on unit and inverse element, because we have learnt it properly since primary school in the subjects of natural numbers, integer numbers, rational numbers, and operations. Even though the concepts are abstract, I don't think it will cause problem if the students comprehend them accurately.

Results and Suggestions

In view of the findings obtained from this research which has been conducted to determine the importance of conceptual and operational learning in linear algebra lesson and the difficulties experienced by the students in the lesson, it is observed that the students are successful in definitions and algorithmic operations, but not successful in the implementations of the definitions. The definitions and operations are defined as the operational information, and the implementations of the definitions are defined as the conceptual information. In line with this, it is seen that conceptual and operational learning are not balanced in the linear algebra lesson: only operational learning exists, and thus the students are unable to implement the concepts and definitions that they learned in linear algebra lesson.

It has been observed that there is no significant difference in the rates of the students' success during their first and latter taking of the course of Linear Algebra II. According to these results, we can report that the students have not learnt such subjects properly as linear transformation, the nucleus of a linear transformation, base, dimension and the coordinations of a vector and the concepts of a matrix which counters to a linear transformation. It is observed that since the students have not grasped the concepts during their first study of the lesson, they are likely to experience difficulties in future terms. The students informed us that they could not learn the subjects conceptually but could only memorize them. In addition, they expressed that they could not learn the definitions conceptually and as a result of this, they got trouble with the questions which require the practicing of those definitions. Reviewing the conducted interviews and the answers given to the questions by the students who have been observed for three years, it can be seen that the reason for the learning difficulties and failures of the students in linear algebra lesson stemmed from the fact that students memorize the concepts related to the subject without knowing their meanings. In the conducted interviews, students stated that they know the definition of linear transformation and the kernel of a linear transformation by heart but they do not know what these definitions mean. In view of these facts, it can be stated that in linear algebra lessons, conceptual and operational learning are not balanced whereas operational learning dominates. The findings obtained from this research have been paralleled to the studies of Harel (1989) and Dreyfus et al. (1998).

According to the answers that the students gave to the questions and the interviews of the students, we can conclude that the students do not have problem with memorizing the concepts about definitions and using the operations correctly, but they get into trouble when they practice the definitions and transfer it to different situations. In this study it has been noticed that the students generally have operational skill in linear algebra. It has been observed that students did not make mistakes in simple algorithmic operations such as calculating the multiplication of two matrixes or the solution of linear equation systems, but they began to make mistakes when it comes to more abstract subjects such as the practice of base, dimension and definitions. Those results are consistent with the studies of Sabella and Redish (1995).

Since the subjects of mathematics are not explained conceptually, the students try to memorize the subjects instead of comprehending them meaningfully. Most of the students in our study sample could not comprehend the definitions and operations they used.

Conceptual learning must take an important place in teaching mathematics. In other words, conceptual and operational learning should be balanced. For teaching mathematics efficiently, it is very significant for teachers to give the definitions of the concepts clearly, to put out the axioms in a clear and straight way and to define the relations of these concepts in a meaningful manner. Moreover the concepts should be shown related to daily life if it is possible.

For an efficient learning of mathematics, the students should learn the definitions, concepts, and structures accurately without memorizing them. In order to apply mathematics teaching efficiently, teachers must focus on finding new methods which will be useful for students to comprehend the concepts about the subjects, to see the operations of the concepts and to set up the relations between concepts and operations.

Mathematics is an abstract subject and methods should be considered to minimizing the difficulties during the process of education. Students often consider mathematics as a scientific subject which is not relevant to both its own branch and to daily life, and as a consequence of this, cognitional and emotional problems appear along with other negative situations. Also the teachers should be able to foresee what kind of difficulties the students will experience when teaching mathematics subjects and try to eliminate these difficulties. The abstractness of mathematics affects the students negatively and as a result the students suppose that they could not learn mathematical concepts and try to memorize the subjects as a solution. The use of efficient methods (giving concrete examples or concreting the models) in teaching and learning of abstract concepts has positive effects on the students from the aspects of cognitional and emotional learning. It is hope that with the help of efficient methods, students are able to understand the meaning of the abstract mathematical concepts, and to comprehend the relations in mathematics more properly.

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