Geometric pmf

- Sequence of Bernoulli,but we count the no. of trials until FIRST SUCCESS.(0=failure,1=success)

$$S = \{0_{i-1}1 \mid i = 1, 2, \cdots\}$$

- Let I be the no. of trials up to and including the first success

$$\begin{aligned}
P_Z(i) &= q^{i-1}p \text{ or } (1-p)^{i-1}p \text{ for } i = 1, 2, \cdots \\
F_Z(t) &= P[Z \leq t] = \sum_{i=1}^{t} p(1-p)^{i-1} = p\sum_{i=1}^{t}(1-p)^{i-1} \\
&= p[\frac{1-(1-p)^t}{1-(1-p)}] \\
&= 1-(1-p)^t
\end{aligned}$$

- For Geometric Distribution $\Longrightarrow$ MEMORYLESS or Markov property

$$P_Z(i) = p(1-p)^{i-1} \text{ for } i = 1, 2, \cdots$$

- Let Z be the r.v representing the no. of trials until the first success. Assume we have observed n trials already and they are all for failure.

- Let Y be the r.v. representing the additional trials until the first success.

$$\begin{aligned}
Y &= Z - n \\
q_i &= P[Y = i \mid Z > n] = P[Z - n = i \mid Z > n] \\
&= P[Z = i + n \mid Z > n] = \frac{P[Z = i + n \;\&\; Z > n]}{P(Z > n)} \\
&= \frac{P[Z = i + n]}{P(Z > n)} = \frac{p(1-p)^{i+n-1}}{(1-p)^n} = p(1-p)^{i-1} = P_Z(i)
\end{aligned}$$

Modified Geometric pmf

- Let X be the r.v. that counts the # of failure before the 1^{st} success.

$$
\begin{aligned}
P_X(i) &= p(1-p)^i \qquad\qquad \text{for } i = 0, 1, 2, \cdots \\
F_X(t) &= \sum_{i=0}^{t} p(1-p)^i = 1 - (1-p)^{t+1}
\end{aligned}
$$

Example : while $(!B)$ $\{S\}$
$(P[B = T] = p; P[B = F] = q = (1-p))$

The # of times S is being executed is M.G.D.

- Defining many R.V. on the sample space(Joint R.V.)
  - A random experiment that assigns values to two r.v. X and Y and a two-dimensional set of values B that corresponds to some of the possible outcomes of the random experiment

$$P[(X,Y) \in B] = P[\{w : (X(w), Y(w)) \in B\}]$$

- **Example:** a random experiment of coin and selection of a ball from an urn

$$\left[\underbrace{B_a}_{0} \quad \underbrace{B_b}_{1} \quad \underbrace{B_c}_{1} \quad \underbrace{B_d}_{2}\right]$$

$$S = \{HB_a, TB_a, HB_b, TB_b, HB_c, TB_c, HB_d, TB_d\}$$

- Let X be the r.v. that counts the no. of heads; Y be the label of the ball.

(X,Y)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$Z^{-1}(X,Y)$	$\{TB_a\}$	$\{TB_b, TB_c\}$	$\{TB_d\}$	$\{HB_a\}$	$\{HB_b, HBc\}$	$\{HB_d\}$
$P[Z(X,Y)]$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

PDF for two R.V.

$$F_{XY}(x,y) = P[X \leq x, Y \leq y] \;\; ; \;\; f_{XY}(x,y) = \frac{d^2 F_{XY}(x,y)}{dxdy}$$

- Marginal Density Function

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)dy$$

- $P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}(xy)dydx$
- $F_{XY}(xy) = F_X(x)F_Y(y) =$ Prob$[X \leq x]$ Prob$[Y \leq y]$ (Independent)
- $F_{X|Y}(x \mid y) = \frac{F_{XY}(x,y)}{F_Y(y)} = \text{Prob}[X \leq x \mid Y \leq y]$

- Impulse Function:

$$\delta_o(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

with the constraint $\int_{-\infty}^{\infty} \delta_o(t) = 1$ or $\int_{-\infty}^{x} \delta_o(t) = \begin{cases} 1 & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

- Example of mixed function
  - Single Server Queueing System

$$\begin{aligned} f_W(x) &= (1-\rho)\delta_o(x) + \rho f_{Z_N}(x) \quad \text{for } x \geq 0 \\ F_W(x) &= (1-\rho) + \rho F_{Z_N}(x) \quad \text{for } x \geq 0 \end{aligned}$$

Function of Random variable

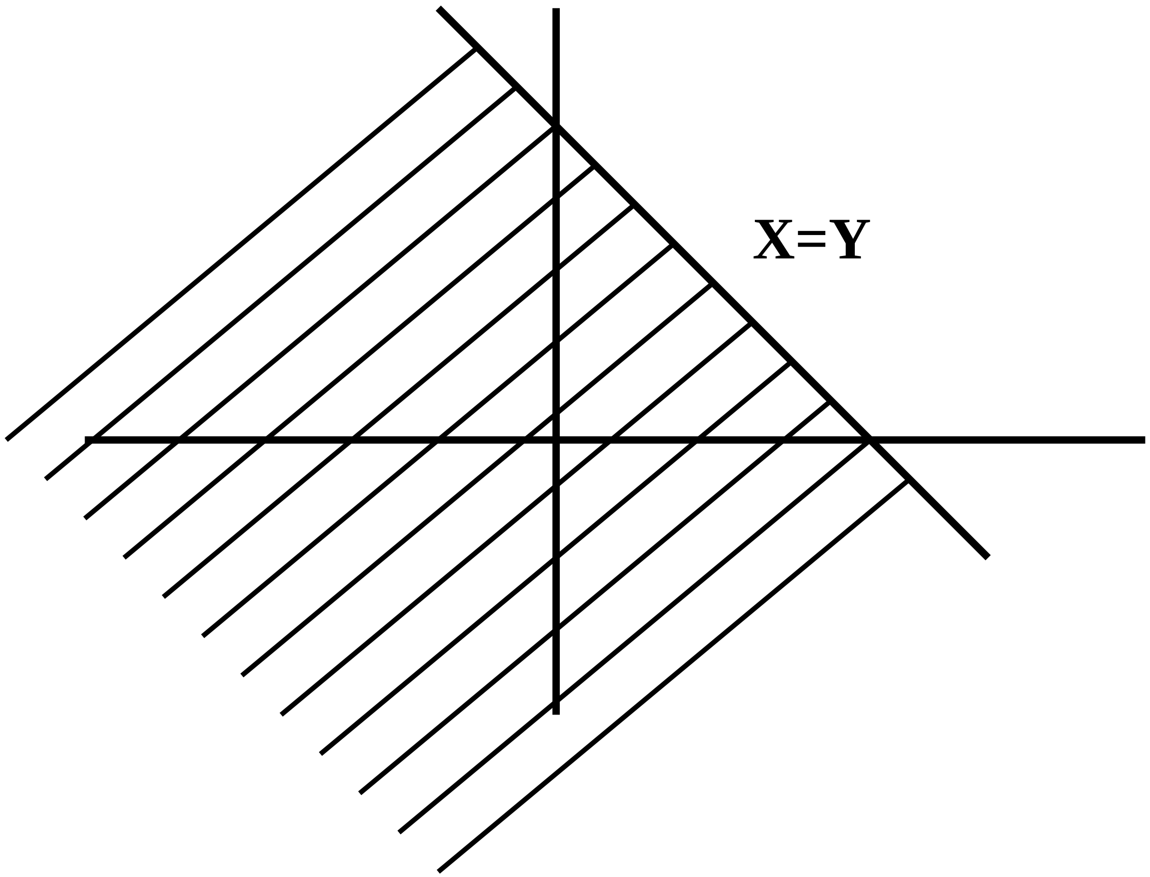
$$Y = g(X)$$

$$F_Y(y) = P[Y \leq y] = P[\{w : g(X(w)) \leq y\}]$$

- One important r.v is where X_i are independent

$$Y = \sum_{i=1}^{n} X_i$$

- example: when $n = 2$, that is $Y = X_1 + X_2$



$$
\begin{aligned}
F_Y(y) &= P[Y \leq y] = P[X_1 + X_2 \leq y] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{y-x_2} f_{X_1X_2}(x_1x_2)dx_1dx_2
\end{aligned}
$$

$$
\begin{aligned}
F_Y(y) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{y-x_2} f_{X_1}(x_1)dx_1 \right] f_{X_2}(x_2)dx_2 \\
&= \int_{-\infty}^{\infty} F_{X_1}(y - x_2) f_{X_2}(x_2)dx_2
\end{aligned}
$$

- Taking the pdf (or pmf)

$$
f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y - x_2) f_{X_2}(x_2)dx_2
$$

$\longrightarrow$ **convolution**

$$
f_Y(y) = f_{X_1}(y) * f_{X_2}(y)
$$

- Therefore, if

$$\begin{aligned} Y &= \sum_{i=1}^{n} X_i \\ f_Y(y) &= f_{X_1}(y) * f_{X_2}(y) \cdots * f_{X_n}(y) \end{aligned}$$

- Note that for convolution, we can use transform

$$G_Y(Z) = G_{X_1}(Z) G_{X_2}(Z) \cdots G_{X_n}(Z)$$

Probability Generating Function

- Let X be the r.v and $G_X(Z)$ be the probability generating function. (Z transform) (Note:X is a non-negative integer value r.v)

$$G_X(Z) = \sum_{i=0}^{\infty} P[X = i] z^i$$

- Why is it useful?
  - Probability distribution $\longleftrightarrow$ probability generating function
  - easier to operate
  - Once we know the probability generating function, we can "crank" out moment of the R.V.

Geometric Sum and Geometric R.V.

- $X = X_1 + X_2 + X_3 + \ldots X_{\tilde{N}}$
- X_i are identical, independent distributed(I.I.D) with Geometric R.V. w/ parameter p and $\tilde{N}$ is a geometric R.V. with parameter α

$$
\begin{aligned}
G_X(Z) &= \sum_{n=1}^{\infty} \left[\frac{pZ}{1-(1-p)Z} \right]^n (1-\alpha)^{n-1}\alpha \\
&= \frac{\alpha p Z}{1-(1-p)Z} \sum_{n=1}^{\infty} \left[\frac{pz(1-\alpha)}{1-(1-p)Z} \right]^{n-1} \\
&= \frac{\alpha p Z}{1-(1-p)Z - pZ(1-\alpha)} \\
&= \frac{\alpha p Z}{1-(1-\alpha p)Z} \rightarrow \text{Geometric distribution}
\end{aligned}
$$

$$
\begin{aligned}
P_X(n) &= \alpha p(1-\alpha p)^{n-1} \\
P_{X_i}(n) &= p(1-p)^{n-1} \\
P_{X_i}(Z) &= \sum_{n=1}^{\infty} p(1-p)^{n-1} Z^n \\
&= \frac{pZ}{1-(1-p)Z}
\end{aligned}
$$

Z-Transform

- Consider a function of discrete time f_n s.t.

$$f_n \geq 0 \qquad \text{for } n = 0, 1, 2, \cdots$$

$$f_n = 0 \qquad \text{for } n = -1, -2, \cdots$$

$f_n \Longleftrightarrow F(Z)$ where $F(Z) = \sum_{n=0}^{\infty} f_n Z^n$

- **Example:**

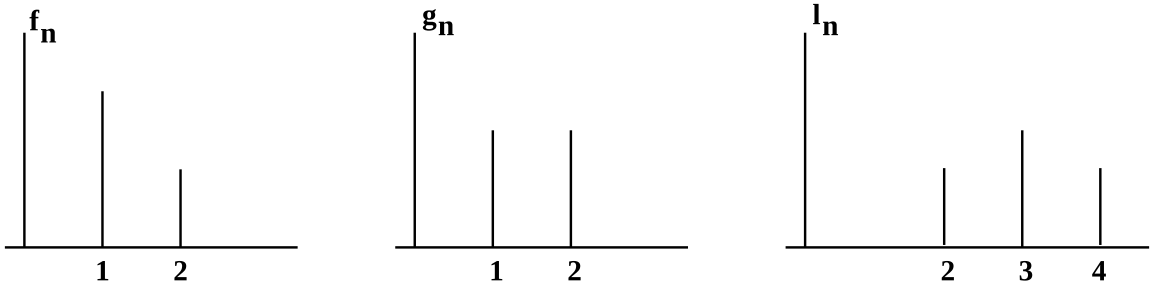
1. Unit impulse function $U_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$

   – Z-transform $F(Z) = \sum_{n=0}^{\infty} U_n Z^n = 1$

2. shift unit function $U_{n-k} = \begin{cases} 1 & n = k \\ 0 & \text{otherwise} \end{cases}$

   – Z-transform $F(Z) = \sum_{n=0}^{\infty} U_{n-k} Z^n = Z^k$

- What's convolution? $f_n * g_n = \sum_{k=0}^{\infty} f_{n-k} g_k = l_n$
- Example:



$$
\begin{aligned}
l_0 &= f_0 g_0 = 0; \qquad\qquad l_1 = f_1 g_0 + f_0 g_1 = 0 \\
l_2 &= f_2 g_0 + f_1 g_1 + f_0 g_2 = \frac{2}{3}\left(\frac{1}{3}\right) = \frac{1}{3} \\
l_3 &= f_3 g_0 + f_2 g_1 + f_1 g_2 + f_0 g_3 = \frac{1}{3}\left(\frac{1}{2}\right) + \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{2} \\
l_4 &= f_4 g_0 + f_3 g_1 + f_2 g_2 + f_1 g_3 + f_0 g_4 = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6}
\end{aligned}
$$

$f_n = A\alpha^n$ for $n = 0, 1, 2, \cdots$

$F(Z) = \sum_{n=0}^{\infty} A\alpha^n Z^n = A\sum_{n=0}^{\infty}(\alpha Z)^n = A[\frac{1}{1-\alpha Z}]$

- Convolution Property $f_n * g_n = \sum_{k=0}^{n} f_{n-k} g_k$

$$
\begin{aligned}
f_n * g_n \Longleftrightarrow \sum_{n=0}^{\infty}[\sum_{k=0}^{n} f_{n-k} g_k] Z^n &= \sum_{n=0}^{\infty}[\sum_{k=0}^{n} f_{n-k} g_k] Z^{n-k} Z^k \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} g_k Z^k f_{n-k} Z^{n-k} \\
&= \sum_{k=0}^{\infty} g_k Z^k \sum_{n=k}^{\infty} f_{n-k} Z^{n-k} \\
&= \sum_{k=0}^{\infty} g_k Z^k \sum_{m=0}^{\infty} f_m Z^m \\
&= G(Z)F(Z)
\end{aligned}
$$

Some Properties of Z transform

1. $f_n \Leftrightarrow F(Z) = \sum_{n=0}^{\infty} f_n Z^n$
2. $af_n + bg_n \Leftrightarrow aF(Z) + bG(Z)$
3. $a^n f_n \Leftrightarrow F(aZ)$
4. $f_{n+1} \Leftrightarrow \sum_{n=0}^{\infty} f_{n+1} Z^n = \frac{1}{Z} \sum_{n=0}^{\infty} f_{n+1} Z^{n+1} = \frac{1}{Z}[\sum_{m=1}^{\infty} f_m Z^m] = \frac{1}{Z}[F(Z) - f_0]$
5. $f_{n-1} \Leftrightarrow \sum_{n=0}^{\infty} f_{n-1} Z^n = Z \sum_{n=0}^{\infty} f_{n-1} Z^{n-1} = Z[\sum_{m=0}^{\infty} f_m Z^m] = ZF(Z)$
6. $nf_n \Leftrightarrow \sum_{n=0}^{\infty} nf_n Z^n = \sum_{n=0}^{\infty} f_n \frac{dZ^n}{dZ} Z = Z \frac{d}{dZ} \sum_{n=0}^{\infty} f_n Z^n = Z \frac{d}{dZ} F(Z)$
7. $f_n * g_n \Leftrightarrow F(Z)G(Z)$

8. $f_n - f_{n-1} \Leftrightarrow F(Z) - ZF(Z) = (1-Z)f(Z)$

9. $\sum_{k=0}^{n} f_k, n = 0, 1, 2, \cdots \Leftrightarrow \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_k Z^n = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} f_k Z^k Z^{n-k} = \frac{F(Z)}{1-Z}$

10. $\frac{\sigma}{\sigma a} f_n \quad (n \text{ is a parameter of } f_n) \;\; \Leftrightarrow \frac{\sigma}{\sigma a} F(Z)$

11. $F(1) = \sum_{n=0}^{\infty} f_n$

12. $F(0) = f_0$

Transform Pair

1. $U_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow F(Z) = 1$

2. $U_{n-k} \Leftrightarrow Z^k$

3. $\delta_n = 1 \quad \text{for } n = 0, 1, \cdots \Leftrightarrow \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

4. $\delta_{n-k} \Leftrightarrow \frac{z^k}{1-z}$

5. $A\alpha^n \Leftrightarrow \sum_{n=0}^{\infty} A\alpha^n z^n = \frac{A}{1-\alpha z}$

6. $n\alpha^n \Leftrightarrow \frac{\alpha z}{(1-\alpha z)^2}$

7. $n \Leftrightarrow \frac{z}{(1-z)^2}$

8. $\frac{1}{n!} \Leftrightarrow F(Z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$
   Read and derive Table I.1 and I.2

Partial Fraction Expansion

$$F(Z) = \frac{N(Z)}{D(Z)} \text{ where } D(Z) = \prod_{i=1}^{k}(1-\alpha_i z)^{m_i} \Leftarrow \text{ meaning?}$$

$$\begin{aligned} F(Z) &= \frac{N(Z)}{D(Z)} = \frac{A_{11}}{(1-\alpha_1 z)^{m_1}} + \frac{A_{12}}{(1-\alpha_1 z)^{m_1-1}} + \ldots + \frac{A_{1m_1}}{(1-\alpha_1 z)} \\ &+ \frac{A_{21}}{(1-\alpha_2 z)^{m_2}} + \frac{A_{22}}{(1-\alpha_2 z)^{m_2-1}} + \ldots + \frac{A_{2m_2}}{(1-\alpha_2 z)} + \ldots + \\ &\frac{A_{k1}}{(1-\alpha_k z)^{m_k}} + \frac{A_{k2}}{(1-\alpha_k z)^{m_k-1}} + \ldots + \frac{A_{km_k}}{(1-\alpha_k z)} \\ A_{ij} &= \frac{1}{(j-1)!}\left(-\frac{1}{\alpha_i}\right)^{j-1} \frac{d^{j-1}}{dz^{j-1}} \left[(1-\alpha_i z)^{m_i} \frac{N(Z)}{D(Z)}\right]\Bigg|_{z=1/\alpha_i} \end{aligned}$$

- Example

$$F(Z) = \frac{4z^2(1-8z)}{(1-4z)(1-2z)^2}$$

Make sure that it's in a proper form (degree of numerator < degree of denominator)

$$\begin{aligned}
F(Z) &= z^2 \underbrace{[\frac{4(1-8z)}{(1-4z)(1-2z)^2}]}_{G(Z)} \\
G(Z) &= \frac{A_{11}}{1-4z} + \frac{A_{21}}{(1-2z)^2} + \frac{A_{22}}{(1-2z)} \\
A_{11} &= (1-4z)G(Z)\mid_{Z=\frac{1}{4}} = \frac{4(1-\frac{8}{4})}{(1-\frac{2}{4})^2} = -16 \\
A_{21} &= (1-2z)^2 G(Z)\mid_{Z=\frac{1}{2}} = \frac{4(1-\frac{8}{2})}{1-\frac{2}{4}} = 12
\end{aligned}$$

$$
\begin{aligned}
A_{22} &= -\frac{1}{2}\frac{d}{dz}[(1-2z)^2 G(Z)] \mid_{Z=\frac{1}{2}} = 8 \\
G(Z) &= \frac{-16}{1-4z} + \frac{12}{(1-2z)^2} + \frac{8}{(1-2z)} \\
G(Z) &= \underbrace{-16(4)^n + 12(n+1)2^n + 8(2)^n}_{g_n} \qquad n = 0,1,2,\cdots
\end{aligned}
$$

$$
f_n = \begin{cases} -16(4)^{n-2} + 12(n-1)2^{n-2} + 8(2)^{n-2} & n = 2,3\cdots \\ 0 & n < 2 \end{cases}
$$

Z transform ⇔ difference equation

$$6g_n - 5g_{n-1} + g_{n-2} = 6(\frac{1}{5})^n \quad n = 2, 3, 4, \cdots ; g_0 = 0; g_1 = \frac{6}{5}$$

Let $G(Z) = \sum_{n=0}^{\infty} g_n z^n$

$6 \sum_{n=2}^{\infty} g_n z^n - 5 \sum_{n=2}^{\infty} g_{n-1} z^n + \sum_{n=2}^{\infty} g_{n-2} z^n = 6 \sum_{n=2}^{\infty} (\frac{1}{5} z)^n$

$6[G(Z) - g_0 - g_1 z] - 5z[G(Z) - g_0] + z^2 G(Z) = 6[\frac{1}{1-\frac{z}{5}} - 1 - \frac{z}{5}]$

$$\begin{aligned}
G(Z) &= (\frac{1}{5}) \frac{z(6-z)}{[1-(\frac{1}{3})z][1-(\frac{1}{2})z][1-(\frac{1}{5})z]} \\
&= \frac{-9}{1-(\frac{1}{3})z} + \frac{8}{1-(\frac{1}{2})z} + \frac{1}{1-(\frac{1}{5})z} \\
g(n) &= -9(\frac{1}{3})^n + 8(\frac{1}{2})^n + (\frac{1}{5})^n
\end{aligned}$$

$n = 0, 1, 2, \cdots \rightarrow$ note $G(1) = \sum_{n=0}^{\infty} g_n = \frac{15}{4}$

Laplace transform

1. $F^*(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$
   Therefore, $f(t) \Leftrightarrow F^*(s)$

2. $f(t) = \begin{cases} Ae^{-at} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} f(t) \Leftrightarrow F^*(s) &= \int_0^{\infty} Ae^{-at}e^{-st}dt \\ &= A\int_0^{\infty} e^{-(s+a)t}dt \\ &= A(-\frac{1}{s+a})e^{-(s+a)t}\ |_0^{\infty} \\ &= \frac{A}{s+a} \end{aligned}$$

3. $\delta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

$$\begin{aligned} \delta(t) \Leftrightarrow F^*(s) &= \int_0^\infty e^{-st} dt \\ &= -\frac{1}{s} e^{-st} \mid_0^\infty \\ &= \frac{1}{s} \end{aligned}$$

4. $U_0(t) = \begin{cases} 1 & t = 0 \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow F^{-1}(s) = \int_0^\infty U_0(t) e^{-st} dt = 1$

Convolution

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx$$

in our case for function w/c takes on non-zero value for $t \geq 0$

$$\begin{aligned}
f(t) * g(t) &= \int_{0}^{t} f(t-x)g(x)dx \\
f(t) * g(t) &\Leftrightarrow \int_{t=0}^{\infty} f(t)*g(t)e^{-st}dt = \int_{t=0}^{\infty} \int_{x=0}^{t} f(t-x)g(x)dx\, e^{-st}dt \\
&= \int_{x=0}^{\infty} \int_{t=x}^{\infty} f(t-x)e^{-s(t-x)}dt\, g(x)e^{-st}dx \\
&= \int_{x=0}^{\infty} g(x)e^{-st}dx \int_{t=x}^{\infty} f(t-x)e^{-s(t-x)}dt = G^*(s)F^*(s)
\end{aligned}$$

Properties:

1. $af(t) + bg(t) \Leftrightarrow aF^*(s) + bG^*(s)$
2. $f(\frac{t}{a}) \overset{?}{\Leftrightarrow} aF^*(as)$
3. $f(t-a) \overset{?}{\Leftrightarrow} e^{-as}F^*(s)$
4. $e^{-at}f(t) \Leftrightarrow F^*(s+a)$
5. $tf(t) \overset{?}{\Leftrightarrow} -\frac{d}{ds}F^*(s)$
6. $t^n f(t) \Leftrightarrow (-1)^n \frac{d^n F^*(s)}{ds^n}$
7. $\frac{f(t)}{t} \overset{H.W.}{\Leftrightarrow} \int_{s_1=s}^{\infty} F^*(s_1)ds_1$

8. $\frac{df(t)}{dt} \Leftrightarrow ?$

First, integration by parts

$$\begin{aligned}
uv &= \int vdu + \int udv \quad \text{or} \quad \int vdu = uv - \int udv \\
F^*(s) &= \int_{t=0}^{\infty} \frac{df(t)}{dt} e^{-st} dt \\
&= f(t)e^{-st} \,|_{t=0}^{\infty} + s\int_{t=0}^{\infty} f(t)e^{-st} dt \\
&= 0 + sF^*(s)
\end{aligned}$$

9. $\frac{d^n f(t)}{dt^n} \Leftrightarrow s^n F^*(s) - s^{n-1} f(0^-) - s^{n-2} f^{(1)}(0^-) - \ldots - f^{(n-1)}(0^-)$

10. $\int_{-\infty}^{t} f(t)dt \overset{H.W.}{\Leftrightarrow} \frac{F^*(s)}{s}$

11. $\int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f(t)(dt)^n \Leftrightarrow \frac{F^*(s)}{s^n}$

Laplace Transform pairs

1. $f(t) \Leftrightarrow F^*(s)$
2. $U_0(t) \Leftrightarrow 1$
3. $U_0(t-a) \Leftrightarrow e^{-as}$
4. $\delta(t) \Leftrightarrow \frac{1}{s}$
5. $Ae^{-at} \Leftrightarrow \frac{A}{s+a}$
6. $te^{-at} \Leftrightarrow \frac{1}{(s+a)^2}$

H.W : re-derive table

Partial fraction Expansion

$$
\begin{aligned}
F^*(s) &= \frac{N^*(Z)}{D^*(Z)} = \frac{B_{11}}{(s+a_1)^{m_1}} + \frac{B_{12}}{(s+a_1)^{m_1-1}} + \ldots + \frac{B_{1m_1}}{(s+a_1)} \\
&+ \frac{B_{21}}{(s+a_2)^{m_2}} + \frac{B_{22}}{(s+a_2)^{m_2-1}} + \ldots + \frac{B_{2m_2}}{(s+a_2)} + \ldots + \\
&\frac{B_{k1}}{(s+a_k)^{m_k}} + \frac{B_{k2}}{(s+a_k)^{m_k-1}} + \ldots + \frac{B_{km_k}}{(s+a_k)} \\
B_{ij} &= \frac{1}{(j-1)!} \frac{d^{j-1}}{ds^{j-1}} \left[(s+a_i)^{m_i} \frac{N(s)}{D(s)} \right] \Bigg|_{s=-a_i}
\end{aligned}
$$

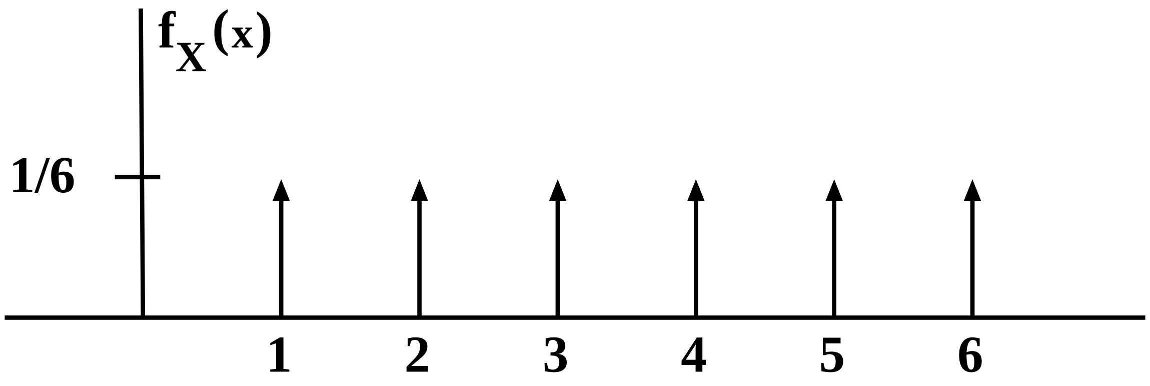
$$
\begin{aligned}
F^*(s) &= \frac{8(s^2+3s+1)}{(s+3)(s+1)^3} = \frac{B_{11}}{s+3} + \frac{B_{21}}{(s+1)^3} + \frac{B_{22}}{(s+1)^2} + \frac{B_{23}}{s+1} \\
B_{11} &= (s+3)F^*(s)\mid_{s=-3} = \frac{8(9-9+1)}{(-2)^3} = -1 \\
B_{21} &= (s+1)^3 F^*(s)\mid_{s=-1} = \frac{8(1-3+1)}{2} = -4 \\
B_{22} &= \frac{d}{ds}\left[\frac{8(s^2+3s+1)}{(s+3)}\right]\mid_{s=-1} = 6 \\
B_{23} &= \frac{1}{2!}\frac{d^2}{ds^2}\left[\frac{8(s^2+3s+1)}{(s+3)}\right]\mid_{s=-1} = 1 \\
F^*(s) &= \frac{-1}{s+3} - \frac{4}{(s+1)^3} + \frac{6}{(s+1)^2} + \frac{1}{s+1} \\
f(t) &= -e^{-3t} - 4t^2e^{-t} + 6te^{-t} + e^{-t} \qquad t \geq 0
\end{aligned}
$$

$\frac{d^2 f(t)}{dt^2} - \frac{6f(t)}{dt} + 9f(t) = 2t \quad ; \quad f(0^-) = 0, \frac{df(0^-)}{dt} = 0$

$s^2 F^*(s) - sf(0^-) - f^{(1)}(0^-) - 6sF^*(s) + 6f(0^-) + 9F^*(s) = \frac{2}{s^2}$

$$
\begin{aligned}
F^*(s) &= \frac{\frac{2}{s^2}}{s^2 - 6s + 9} = \frac{2}{s^2(s-3)^2} \\
&= \frac{\frac{2}{9}}{s^2} + \frac{\frac{4}{27}}{s} + \frac{\frac{2}{9}}{(s-3)^2} + \frac{\frac{-4}{27}}{s-3} \\
f(t) &= \frac{2}{9}t + \frac{4}{27} + \frac{2}{9}te^{3t} - \frac{4}{27}e^{3t} \qquad t \geq 0
\end{aligned}
$$

- Example : given a pmf of a R.V. X



What's the "average outcome"?

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- Expectation of a R.V. X (assuming X is a discrete R.V.)

$$\bar{X} = E[X] = \sum_{n=0}^{\infty} n P_X(n)$$

- If R.V X is continuous

$$\begin{aligned} \bar{X} = E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x dF_X(x) \\ &= \int_0^{\infty} [1 - F_X(x)] dx - \int_{-\infty}^{0} F_X(x) dx \end{aligned}$$

$(F_X(\infty) = 1, F_X(-\infty) = 0)$

- if X is a non-negative R.V. then $[xF_X(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} F_X(x)dx$

$$\bar{X} = E[X] = \int_0^{\infty} [1 - F_X(x)] dx$$

$$
\begin{aligned}
E[X+Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f_{XY}(xy)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(xy)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(xy)dxdy \\
&= \int_{-\infty}^{\infty} x f_X(x)dx + \int_{-\infty}^{\infty} y f_Y(y)dy \\
&= E[X]+E[Y]
\end{aligned}
$$

Expectation of sum of R.V is equal to the sum of each individual expectation.

$$
E[X_1+X_2+X_3+\ldots+X_n] = E[X_1]+E[X_2]+\ldots+E[X_n]
$$

$$E[XY] = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} xy f_{XY}(x,y) dx dy$$

If X and Y are independent, we have:

$$\begin{aligned} E[XY] &= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} xy f_X(x) f_Y(y) dx dy \\ &= \int_{x=-\infty}^{\infty} x f_X(x) dx \cdot \int_{y=-\infty}^{\infty} y f_Y(y) dy \\ &= E[X]E[Y] \end{aligned}$$

The n^{th} moment of X is:

$$\begin{aligned} E[X^n] = \bar{X^n} &= \int_{-\infty}^{\infty} x^n f_X(x) dx \quad \text{for continuous case} \\ &= \sum_{l=0}^{\infty} l^n P_X(l) \qquad \text{for discrete case} \end{aligned}$$

variance: $\sigma_X^2 = \bar{X^2} - (\bar{X})^2$

coefficient of variation $= C_X = \sigma_X / \bar{X}$

From Transform $\Longrightarrow$ Moment

Laplace Transform : $F^*(s) = \int_0^\infty f(t)e^{-st}dt$

$$\frac{dF^*(s)}{ds} = \int_0^\infty f(t)\frac{de^{-st}}{ds}dt = \int_0^\infty f(t)(-t)e^{-st}dt$$

$$\left.\frac{dF^*(s)}{ds}\right|_{s=0} = -\left.\int_0^\infty tf(t)e^{-st}dt\right|_{s=0}$$

$$= -\int_0^\infty tf(t)dt = -\bar{X}$$

$$\left.\frac{d^2F^*(s)}{ds^2}\right|_{s=0} = \left.\int_0^\infty t^2f(t)e^{-st}dt\right|_{s=0} = \bar{X^2}$$

In general, we have: $\bar{X^n} = (-1)^n \left.\frac{d^nF^*(s)}{ds^n}\right|_{s=0} = (-1)^n \frac{d^nF^*(0)}{ds^n}$

- Example

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{X} = \int_0^\infty \underbrace{x}_{u} \underbrace{\lambda e^{-\lambda x}}_{dv} dx \quad , \quad \int udv = uv - \int vdu$$

$$\begin{aligned} \bar{X} &= x(-e^{-\lambda x}) \mid_0^\infty - \int_0^\infty -e^{-\lambda x} dx \\ &= [0 - 0] + \int_0^\infty e^{-\lambda x} dx \\ &= -\frac{1}{\lambda} e^{-\lambda x} \mid_0^\infty \\ &= -[0 - \frac{1}{\lambda}] \\ &= \frac{1}{\lambda} \end{aligned}$$

Moment generation using transform

$$
\begin{aligned}
F^*(s) &= \frac{\lambda}{s+\lambda} \\
\bar{X} &= (-1)\frac{dF^*(s)}{ds}\;|_{s=0} \\
&= (-1)[\frac{(s+\lambda)0-\lambda}{(s+\lambda)^2}]\;|_{s=0} \\
&= \frac{\lambda}{(s+\lambda)^2}\;|_{s=0} = \frac{1}{\lambda} \\
\bar{X^2} &= \frac{d^2F^*(s)}{ds^2}\;|_{s=0} = \frac{\lambda(2)(s+\lambda)}{(s+\lambda)^4}\;|_{s=0} \\
&= \frac{2\lambda^2}{\lambda^4} = \frac{2}{\lambda^2} \\
\sigma^2 &= \bar{X^2} - (\bar{X})^2 = \frac{2}{\lambda^2} - (\frac{1}{\lambda})^2 = \frac{1}{\lambda^2}
\end{aligned}
$$

Z Transform $\Rightarrow$ moment

$$
\begin{aligned}
G(Z) &= \sum_{n=0}^{\infty} f_n Z^n \\
\frac{dG(Z)}{dZ} &= \sum_{n=0}^{\infty} n f_n Z^{n-1} \\
\left.\frac{dG(Z)}{dZ}\right|_{z=1} &= \sum_{n=0}^{\infty} n f_n = \bar{X} \\
\frac{d^2G(Z)}{dZ^2} &= \sum_{n=0}^{\infty} n(n-1) f_n Z^{n-2} \\
\left.\frac{d^2G(Z)}{dZ^2}\right|_{z=1} &= \sum_{n=0}^{\infty} (n^2-n) f_n = \bar{X^2} - \bar{X}
\end{aligned}
$$

Random Sum

$$Y = X_1 + X_2 + \ldots + X_{\tilde{N}}, \quad \text{find } Y^*(s)$$

X_i are identical, independent distributed. $X_i^*(s)$ are the same, call it $X^*(s)$. $\tilde{N}$ is a discrete non-negative R.V.

$$\begin{aligned} Y^*(s) &= \sum_{n=0}^{\infty} Y^*(s \mid \tilde{N} = n)\text{Prob}\,[\tilde{N} = n] \\ Y^*(s \mid \tilde{N} = n) &= [X^*(s)]^n \\ Y^*(s) &= \sum_{n=0}^{\infty} [X^*(s)]^n \text{Prob}[\tilde{N} = n] \end{aligned}$$

Note that $N(Z) = \sum_{n=0}^{\infty} \text{Prob}[\tilde{N} = n] Z^n$. Relate Z with $[X^*(s)]^n$, we have:

$$Y^*(s) = N(X^*(s))$$

- Example : $Y = X_1 + X_2 + \ldots X_{\tilde{N}}$
  X_i are i.i.d exponential distributed, $\tilde{N}$ is geometrically distributed.
  $f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0; \quad P_N(n) = p(1-p)^{n-1} \quad n = 1, 2, \cdots$
  $F_X^*(s) = \int_{x=0}^{\infty} \lambda e^{-\lambda x} e^{-sx} dx = \frac{\lambda}{s+\lambda};$
  $N(Z) = \sum_{n=1}^{\infty} p(1-p)^{n-1} Z^n = \frac{p}{1-p} \sum_{n=1}^{\infty} [(1-p)Z]^n$
  $= \frac{p}{1-p}[\frac{(1-p)Z}{1-(1-p)Z}] = \frac{pZ}{1-(1-p)Z}$
  $\bar{X} = -\frac{d}{ds} F^*(s) \mid_{s=0} = (-1)[\frac{-\lambda}{(s+\lambda)^2}] \mid_{s=0} = \frac{1}{\lambda};$
  $\bar{N} = \frac{d}{dZ} N(Z) \mid_{Z=1} = \frac{(1-q)p+pq}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$
  $Y^*(s) = N^*(F^*(s)) = \frac{p\frac{\lambda}{s+\lambda}}{1-q\frac{\lambda}{s+\lambda}}$
  $Y^*(s) = -\frac{d}{ds} Y^*(s) \mid_{s=0} = \frac{1}{\lambda}\frac{1}{p}$
  HW:In general, what's σ_Y^2 ?