CSC 5420

Computer System Performance Evaluation

- Grading: Open for discussion.
- Allocate Four Session per Week
- TEXTBOOK:
  1. **Queueing Systems : Vol I** by Kleinrock
  2. **P.E.** by Peter Harrison
  3. **Introduction to Probability and Statistics** by S.Ross
  4. **Probability, Stochastic Processes, and Queueing Theory**, by Randy Nelson, Springer-Verlag, 1995.

Materials

1. Probability
2. Stochastic Process (Markov Process)
3. Single Server Queueing Theory
   - M/M/1, M/G/1, G/M/1, G/G/1
4. Priority Queueing
5. Multiple Servers queueing theory
6. Queueing Network
   - Product-form
   - Computational algorithms
   - Reversibility
7. Stochastic Comparison
8. Matrix-Geometric Method

9. Large Deviation Theory
10. Simulation & Numerical Methods
11. Transient Analysis

Probability

1. Sample Space (S) which is a collection of objects. Each object is a sample point.
2. A family of event $\Sigma = \{A, B, C, \ldots\}$ where a event is a set of sample points.
3. A Probability measure P is an assignment (mapping) of events defined on S into the set of real numbers.

$$P[A] = \text{Probability of event } A$$

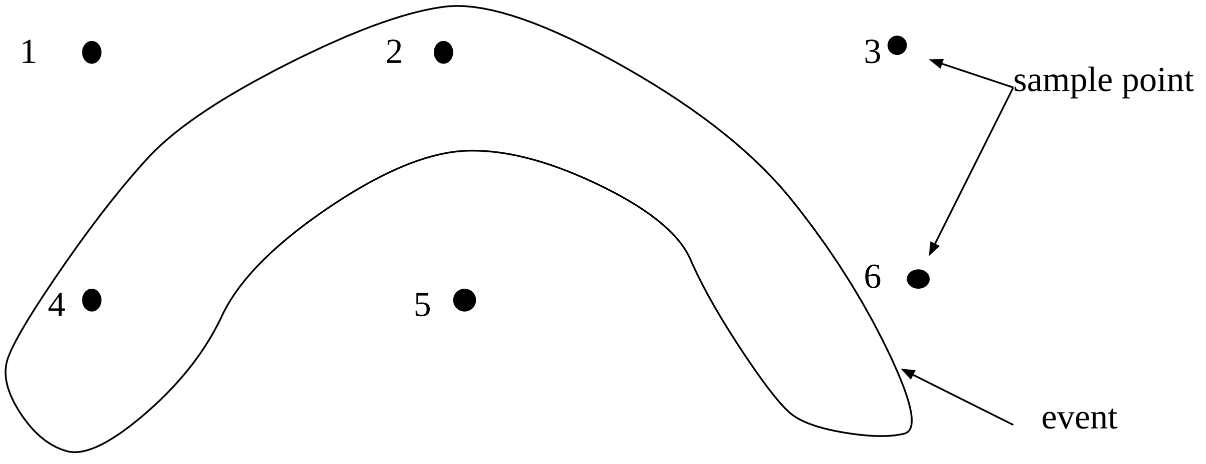
Properties:

1. $0 \leq P[A] \leq 1$
2. $P[S] = 1$
3. If A and B are mutually exclusive, then

$$P[A \cup B] = P[A] + P[B]$$

Notation:

- Event
  $A = \{w : w \text{ satisfies the membership property for the event } A\}$
- Example: dice



- A^c is the complement of A where $\{w : w \text{ not in } A\}$
- $A \cup B = \{w : w \text{ in } A \text{ or } B \text{ or both}\} = \text{UNION}$
- $A \cap B = \{w : w \text{ in } A \text{ and } B\ \} = \text{INTERSECTION}$
- $\emptyset$ = empty set

In general:

- $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

- Proof:

$$
\begin{aligned}
P[A \cup B] &= P[A] + P[A^c \cap B] && (1) \\
P[B] &= P[A \cap B] + P[A^c \cap B] && (2) \\
\text{So, } P[A \cup B] &= P[A] + P[B] - P[A \cap B]
\end{aligned}
$$

- Let $A_1, A_2, \cdots A_n$ be any events

$$
\begin{aligned}
P[\cup_{i=1}^{n} A_i] = & \sum P[A_i] - \sum_{1 \le i < j \le n} P[A_i \cap A_j] + \\
& \sum_{1 \le i < j < k \le n} P[A_i \cap A_j \cap A_k] + \cdots + \\
& (-1)^{n-1} P[A_1 \cap A_2 \ldots \cap A_n]
\end{aligned}
$$

CONDITIONAL PROBABILITY

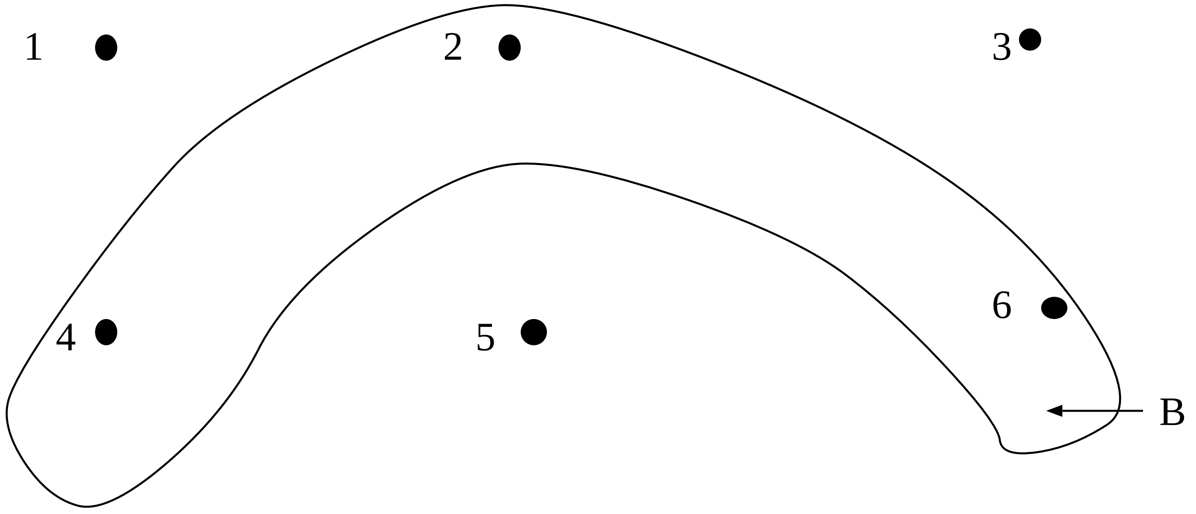
$$P[A|B] = \frac{P[AB]}{P[B]} \quad \text{whenever} \quad P[B] \neq 0$$

$\Longrightarrow$ **CONSTRAINT SAMPLE SPACE**

$\Longrightarrow$ **SCALE UP**

$$P[s|B] = \begin{cases} \frac{P[s]}{P[B]} & \text{if } s \in B \\ 0 & \text{otherwise} \end{cases}$$

- Example



- $\text{Prob}[x \leq 4|B] = \frac{2}{3} = \frac{\frac{1}{6}+\frac{1}{6}}{\frac{1}{2}}$

Example :

A box with 5000 chips, 1000 from company X, other from Y.

10% from X is defective, 5% from Y is defective.

A="chip is from X", B="chip is defective"

$P[A|B] =? \frac{P(AB)}{P(B)}$

- What's the sample point?
- What's their probability?

$$
\begin{aligned}
P[B] &= \frac{1000 \times 0.1 + 4000 \times 0.05}{5000} = 0.06 \\
A \cap B &= \text{chip made by X and it is defective} \\
P[A \cap B] &= \frac{100}{5000} = 0.02 \\
P[A \mid B] &= 0.02/0.06 = \frac{1}{3}
\end{aligned}
$$

STATISTICAL INDEPENDENT (S.I.)

If A and B are statistically independent, then $P[A \cap B] = P[A]P[B]$
if $A_1, A_2, \cdots$ are S.I. $\Longrightarrow$
$P[A_1 \cap A_2 \cap \cdots \cap A_n] = \Pi_{i=1}^{n} P[A_i]$
Also, if S.I, then:

$$P[A \mid B] = \frac{P[AB]}{P[B]} = P[A]$$

physically,what does this mean?

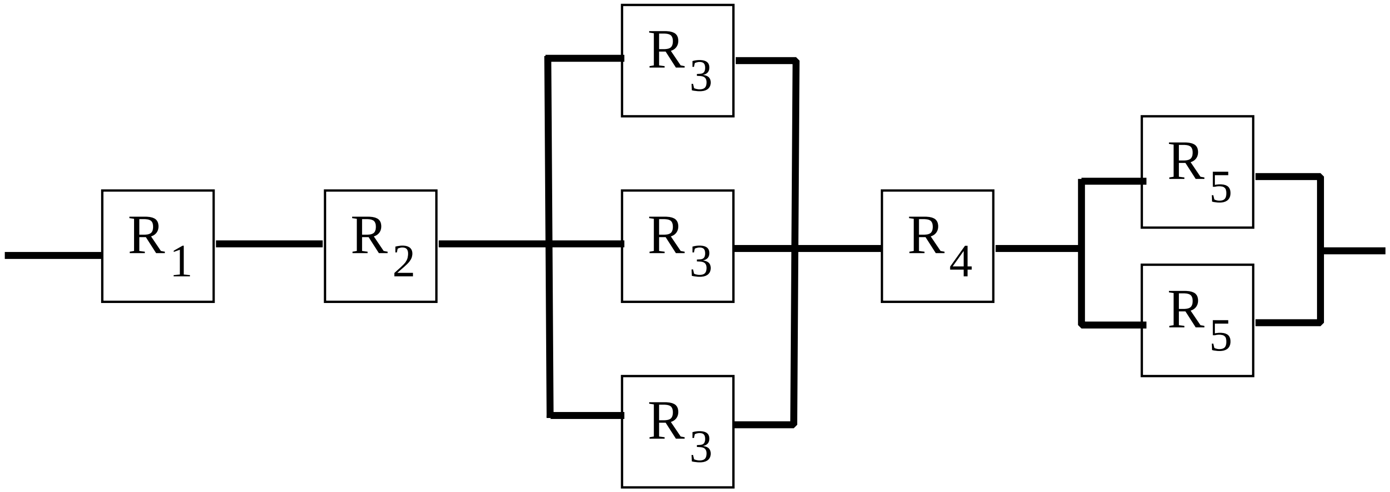
Theorem of total Probability

$$P[B] = \sum_{j=1}^{n} P[B \mid A_i]P[A_i]$$

where $\{A_i\}$ is a set of mutually exclusive exhaustive events.

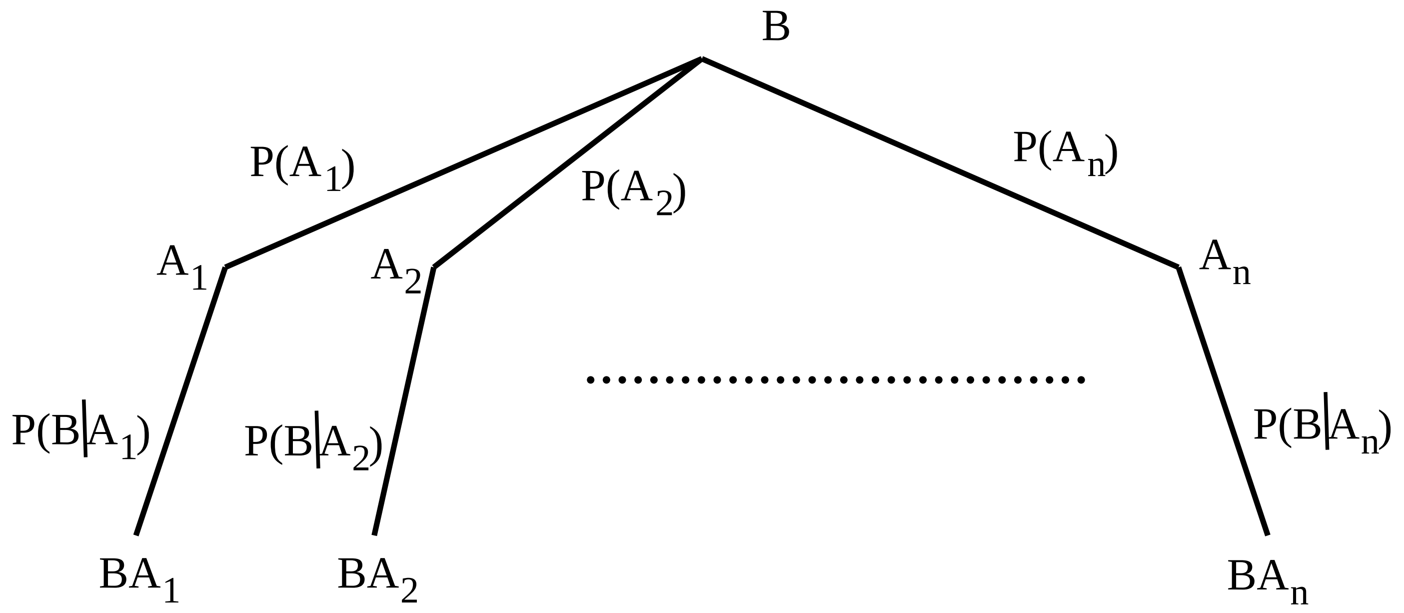
Application of S.I.

R_i = reliability of component i



$$R_{sys} = R_1 * R_2 * (1 - (1 - R_3)^3) * R_4 * (1 - (1 - R_5)^2)$$

- **The importance of the theorem of total Probability is to break up a "complex problem" into many "simpler problems".**



Bayes' Theorem

- Bayes' Theorem : We can look at the problem from another perspective. Assume we know event B has occurred, but we want to find which mutually exclusive event has occurred.

$$P[A_i \mid B] = \frac{P[B \mid A_i]P[A_i]}{\sum_{i=1}^{n} P[B \mid A_i]P[A_i]}$$

- Example :
  Assume That 15% of job is from I.E., 35% of job is from E.E., 50% of job is from C.S&E. Probability of read news are $0.01, 0.05, 0.02$ respectively.
  P[a job chosen in random is a Read news job]
  P[a randomly chosen job is from EE $\mid$ it is a read news job]

- application of theorem of total probability:

$$\begin{aligned} P(A) &= 0.01(0.15) + 0.05(0.35) + 0.02(0.5) \\ &= 0.029 \end{aligned}$$

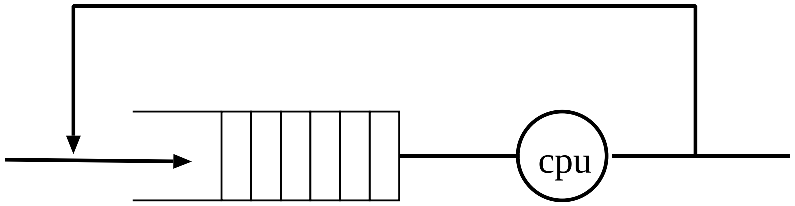
- application of Bayes' rule

$$P[B_2 \mid A] = \frac{(0.05)(0.35)}{0.029} = 0.603$$

- Another example (LK page 367)
  D_H = Probability you play with honest player
  D_C = Probability you play with cheating player
  P = Probability you lose when play with cheating player

$$
\begin{aligned}
P[D_c \mid L] &= \frac{P(D_c \cap L)}{P(L)} = \frac{P(L \mid D_c)P(D_c)}{P(L)} \\
&= \frac{P(L \mid D_c)P(D_c)}{P(L \mid D_c)P(D_c) + P(L \mid D_H)P(D_H)} \\
&= \frac{p(\frac{1}{2})}{p(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2})} = \frac{2p}{2p+1}
\end{aligned}
$$

- **Bernoulli trials** : A random experiment that has two outcomes,"success" (p) or "failure" (q). Now consider a **compound** sequence of n independent repetitions of this experiment. This is known as sequence of Bernoulli trial.
- applications:
  - while loop
  - assembly line checking status of component
  - transmission of K bits file



- Probability of exactly k success after n trials

$$P(k) = C_k^n p^k (q)^{n-k}$$

Verification:

$$\sum_{k=0}^{n} P(k) = 1?$$

$$\begin{aligned}\sum_{k=0}^{n} C_k^n \, p^k q^{n-k} &= (p+q)^n \leftarrow \text{binomial theorem} \\ &= 1\end{aligned}$$

Example: A communication channel with p being the probability of successful transmission of a bit. Assume we design a code that can tolerate up to e bit errors w/n bit word code.
Prob[successful word transmission] =?

$$
\begin{aligned}
P_N &= P[\ e \text{ or fewer errors in } n \text{ trials}] \leftarrow \text{(what assumption made?)} \\
&= \sum_{i=0}^{e} C_i^n (1-p)^i p^{n-i}
\end{aligned}
$$

P.S.: If no parity of checking, $e = 0$, single error-correction Hamming code, $e = 1$

- Generalization : sequence of n independent trials,each trial the result is **EXACTLY** one of the k possibilities, $b_1, \cdots b_k$ with probability $p_1, p_2, \cdots p_k$ s.t

$$p_i \geq 0, \sum_{i=1}^{k} p_i = 1$$

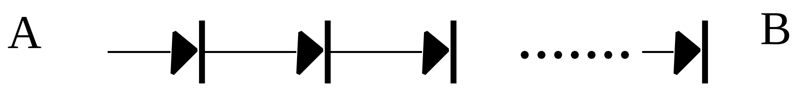
- Let $s \in S$ (sample space)

$s = (\underbrace{b_1 b_1 \cdots b_1}_{n_1}, \underbrace{b_2 b_2 \cdots b_2}_{n_2}, \cdots, \underbrace{b_k b_k \cdots b_k}_{n_k})$

such that: $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ and $\sum_{i=1}^{k} n_i = n$

$P(n_1, n_2, \cdots, n_k) = \frac{n!}{n_1! n_2! \cdots n_k!} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$

Example: A diode can function properly (p_1),short circuit (p_2) or open circuit (p_3). Assume I have the following circuit,what's the probability it will work properly?



$$
\begin{aligned}
&= \sum_{n_1 \geq 1, n_2 \geq 0, n_1+n_2=n} p(n_1, n_2, 0) \\
&= \sum_{n_1 \geq 1, n_2 \geq 0, n_1+n_2=n} \binom{n}{n_1, n_2, 0} p_1^{n_1} p_2^{n_2} p_3^0 \\
&= \sum_{n_1 \geq 1} \frac{n!}{n_1!(n-n_1)!} p_1^{n_1} p_2^{n-n_1} \\
&= \sum_{n_1=0}^{n} C_{n_1}^{n} p_1^{n_1} p_2^{n-n_1} - \frac{n!}{0!n!} p_1^0 p_2^n = (p_1+p_2)^n - p_2^n = (1-p_3)^n - p_2^n
\end{aligned}
$$

- Example: To increase the probability of correct transmission of a message over noisy channel, a **repetitive** code is used.
  Let us assume a message is a single bit and p is the probability of success transmission.
  Repetitive code of rate $\frac{1}{n}$ means the message is transmitted n times and a "majority rate" at the receiver.

- Let us assume $n = 2m - 1$ where $m = 1, 2, \cdots$ what is the $P_c = \text{Prob[error probability]}$?
  $n = 1, m = 1$
  $n = 3, m = 2$
  $$\underbrace{- \; - \; - \; - \; - \; - \; \cdots \; -}_{n}$$
  $n = 5, m = 3$
  $\vdots$

- Let C_n =no. of times of message transmitted correctly out of n trials= $C_k^n p^k (1-p)^{n-k}$ and P_s is the probability of successful transmission.

$$
\begin{aligned}
P_s &= \sum_{k=m}^{n} C_k^n p^k (1-p)^{n-k} \\
P_c &= \sum_{k=0}^{m-1} C_k^n p^k (1-p)^{n-k}
\end{aligned}
$$

If B then

    repeat S_1 until B_1

else

    repeat S_2 until B_2

- Let $P[B = \text{true}] = p$, $P[B_1 = \text{true}] = \frac{3}{5}$, $P[B_2 = \text{true}] = \frac{2}{5}$
- Exactly one statement is common to statement group S_1 and S_2: printf("Hello").
- After a very long experiment,we have $P[\text{exactly 3 "Hello"}] = \frac{3}{25}$. What is p?
- Let A be the event of intersect

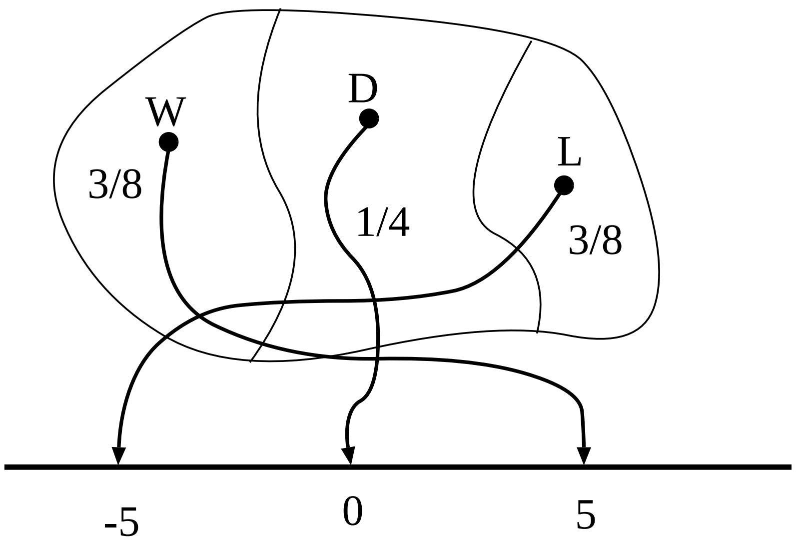
$$
\begin{aligned}
P[A] &= P[B]P[B_1^c]^2 P[B_1] + P[B^c]P[B_2^c]^2 P[B_2] = \frac{3}{25} \\
&= p(\frac{2}{5})^2(\frac{3}{5}) + (1-p)(\frac{3}{5})^2(\frac{2}{5}) = \frac{3}{25}
\end{aligned}
$$

Solve for $p = 0.5$

RANDOM VARIABLES (R.V.)

- We have the probability system (S, Σ, P)
- R.V. is a variable whose value depends upon the outcome of the random experiment.
- The OUTCOMES of our random experiment is a $w \in S$. We associate a real number $X(w)$ to w.
- Thus our r.v $X(w)$ is nothing more than a function defined on the sample space S.

Example:Playing a game is L.Vegas



Notation: R.V. X **on a sample space** S

$$X : S \longrightarrow R$$

Example: Random experiment of three Bernoulli trials. The sample consists of eight triples of 0's and 1's.

Sample points	P(s)	X(s)
111	0.125	3
110	0.125	2
101	0.125	2
100	0.125	1
011	0.125	2
010	0.125	1
001	0.125	1
000	0.125	0

Why we want to study random variables? Consider a singer server FCFS queue with infinite waiting room. Let

- S_n = service time of the n^{th} customer
- A_n = interarrival time between the n^{th} and $(n-1)^{th}$ customer
- W_n = waiting time of the n^{th} customer
- T_n = response time of the n^{th} customer

Experiment: selecting a sample point from an infinite set of $\{S_n, A_n\}$

$$\begin{aligned} T_n &= W_n + S_n \quad n = 1, 2, \ldots \\ W_n &= (T_{n-1} - A_n)^+ = (W_{n-1} + S_{n-1} - A_n)^+ \quad n = 1, 2, \ldots \end{aligned}$$

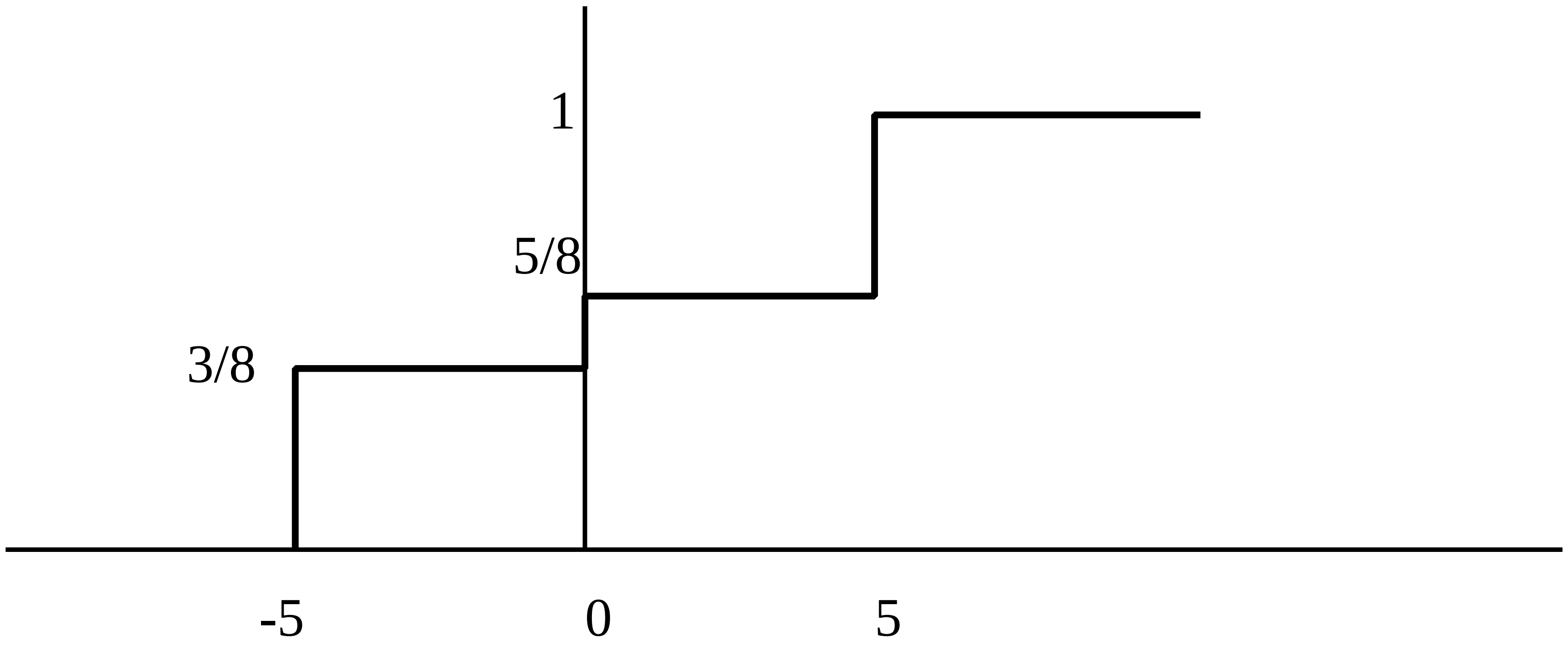
Probability distribution function(PDF)

or cumulative distribution

$$[X \leq x] = \{w : X(w) \leq x\}$$

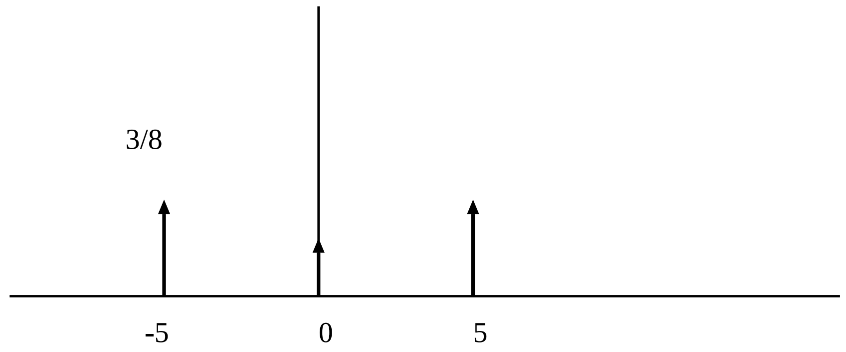
PDF is defined as $F_X(x) = P[X \leq x]$
Properties:

1. $F_X(x) \geq 0$
2. $F_X(\infty) = 1$
3. $F_X(-\infty) = 0$
4. $F_X(b) - F_X(a) = P[a < X \leq b]$ for $a < b$
5. $F_X(b) \geq F_X(a)$ for $a \leq b$


1
5/8
3/8
-5
0
5

Probability density function (pdf) or (pmf)

$$f_x = \frac{dF_X(x)}{dx}$$



Different ways to view pdf:

1. $F_X(x) = \int_{-\infty}^{x} f_X(y)dy$
2. $f_X(x) \geq 0$
3. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
4. $P[a < X \leq b] = \int_a^b f_X(x)dx$

Special discrete distribution

- The Bernoulli pmf

$$
\begin{aligned}
P_X(0) &= P[X = 0] = q \\
P_X(1) &= P[X = 1] = p = (1 - q)
\end{aligned}
$$

- The Binomial pmf: n independent Bernoulli trials

$$
\begin{aligned}
P_k &= C_k^n p^k (1-p)^{n-k} \text{ for } 0 \leq k \leq n \\
P[X \leq x] &= \sum_{k=0}^{x} p_k = \sum_{k=0}^{x} C_k^n p^k (1-p)^{n-k}
\end{aligned}
$$

Geometric pmf

- Sequence of Bernoulli,but we count the no. of trials until FIRST SUCCESS.(0=failure,1=success)

$$S = \{0_{i-1}1 \mid i = 1, 2, \cdots\}$$

- Let I be the no. of trials up to and including the first success

$$\begin{aligned}
P_Z(i) &= q^{i-1}p \text{ or } (1-p)^{i-1}p \text{ for } i = 1, 2, \cdots \\
F_Z(t) &= P[Z \leq t] = \sum_{i=1}^{t} p(1-p)^{i-1} = p\sum_{i=1}^{t}(1-p)^{i-1} \\
&= p[\frac{1-(1-p)^t}{1-(1-p)}] \\
&= 1-(1-p)^t
\end{aligned}$$